Credit Migration of an Internal Rating System for a Canadian SME Loans Portfolio

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Abstract

Credit Migration of an Internal Rating System for a Canadian SME Loans Portfolio

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In the context of Basel II, credit migration matrices have become major components in credit risk management. This thesis is an empirical investigation of various issues that arise in the analysis of credit rating migration. Using the unique and rich internal rating data of a Canadian SME loan portfolio, the thesis investigates credit rating migration from different perspectives. More specifically, the thesis compares and analyzes three multi-state Markov chain estimation methods of migration matrices to accommodate the special features of the internal rating data. The drivers of differences among the three competing methods are investigated theoretically. The statistical and economical differences are examined through non-parametric bootstrap and credit VaR techniques respectively. The underlying assumptions of the three competing methods, time homogeneous first-order Markov chain, are then examined. In the final chapter, the thesis adopts a one-factor model and applies it to our unique internal rating data to forecast migration matrices conditioning on macroeconomic variables. The forecasts are based on different estimation methods of migration matrices. We illustrate the extent to which a one-factor migration matrix could improve the accuracy of the credit loss distribution. Then hypothetical scenarios are selected to evaluate stressed migration matrices with corresponding stressed economic capital.

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Chapter 1

Overview

This thesis is an empirical investigation of various issues that arise in the analysis of credit rating migration. Using the internal rating database of a Canadian financial institution, the thesis investigates credit rating migration from different perspectives. More specifically, we contrast various estimation methodologies for estimating the institution's rating migration; and we apply our results in a forecasting exercise with particular emphasis on stress-testing.

Credit risk, which is the risk that an obligor defaults and does not honor its obligation to service debt, is the most immediate risk faced by a bank in its traditional intermediation functions. Credit ratings are the foundation for managing the credit risk of the banks' loan portfolio. Rating assessments may evolve across time and cause the ratings to change. Rating migration or transition matrices, which reflect the past changes in credit quality of obligors (typically firms), represent the probability of moving from one rating class to another within a given amount of time. It is important to measure changes in obligor credit quality as obligor risk ratings are a key component of a banks' credit capital determination. These analyses permit banks to assess and price credit risk more accurately, as well as improve their assessment of portfolio capital requirements.

In the context of the Basel II requirements concerning the implementation of an internal rating based (IRB) approach, studies on the topics of internal credit rating and corresponding rating migration have emerged in recent years both in the academic and professional industries. In reviewing the literature, we can distinguish several important challenges:

- establishing the proper methodology for estimating migration matrices to assess more accurately capital requirements;

- evaluating forecasting models for the probability of default and rating migration so as to evaluate stress scenarios.

In essence, there are three competing methods for estimating migration matrices: the industry standard "Cohort" method, the academically well known "Duration" method and the "Mixed Time Duration "(MT-Duration) method. These are all multi-state Markov chain models, but with different time settings and data observation patterns. Although the underlying assumptions of first-order and time homogeneity for the three methods may be difficult to maintain over a long horizon (Bangia et al., 2002; Mählmann, 2006), the existing literature offers few practical alternatives.

As for the second challenge to develop procedures for forecasting ratings migrations, the academic literature has used factor models to build an unobserved credit indicator to model the migration matrix in different (structural or reduced form) frameworks (Kim, 1999; Wei, 2003; Bae and Kulperger, 2009; Berteloot et al., 2013). Another major technique for modeling migration matrix involves numerical adjustment methods Jarrow et al. (1997); Lando (2000) based on the macroeconomic situation. Though this method is convenient and is used by financial institution, its accuracy has been questioned in Trück (2008).

It is worth noting that most of the rating migration studies in the academic literature are based on external ratings (i.e. Moody's or Standard& Poor's) database. Due to the scarcity of data, only several papers have used the internal rating system of a financial institution. Moreover, there have been few studies that have focused on small commercial loans, despite the relatively high share of SME exposures in banks' loan portfolio. By contrast, this thesis uses a unique and rich internal rating data associated with a Canadian SME loans portfolio, covering 59,701 Small and Medium Enterprise (SME) obligors and a time period from 1998 to 2009. The SME portfolio is not investment grade and contains a considerable default and migration history.

Within this general framework, this thesis makes the following contributions:

(A) *A comprehensive analysis of an internal rating system for a SME loans portfolio* - SME portfolios in literature have limited historical time spans. Examples of internal rating systems from a SME portfolio are rare. Our comprehensive analysis of the internal rating data associated with a Canadian *Financing Company*'s SME loans portfolio from 1998 to 2009 provide a unique database on which significant analyses can be performed.

(B) *The first implementation of credit migration estimation based on the internal rating data of a SME loan portfolio* - The availability of our unique data is an important contribution to the analysis of credit migrations that emphasizes the internal ratings of a SME loan portfolio. Our internal rating data exhibits irregular pattern of observation times in so far as the rating review policy depends on obligor-specific characteristics. This feature contrasts with the external and internal ratings data in the existing literature.

(C) *The first comparison between Duration and MT-duration estimation methods of migration matrices* - As a key input of credit risk management, an accurately estimated migration matrix associated with internal ratings becomes important. Mählmann (2006) proposed an intensity-based MT-Duration to accommodate interval-censored transition times and the absence of information concerning the censored-states. The paper only compared the MT-Duration with the Cohort method, which leads to a natural question of the difference between the Duration and MT-Duration methods, since they are both undertaken in a continuous time setting. Our comparative analysis is a first contribution in this area.

(D) *The first implementation of a conditional approach to forecast migration matrices based on SME internal ratings* - There have been several studies to estimate migration matrices conditional on macroeconomic variables to highlight rating changes over the business cycle. However, existing research is based on external rating data. We implement it for the first time for the SME internal ratings based on both the Cohort migration and the Duration migration matrices.

(E) *The feasibility of the one-factor model for forecasting migration matrices* - As we determine via an analysis of in-sample and out-of-sample performance, the one-factor model using the internal rating data of the SME loan portfolio is appropriate for forecasting credit migration matrices conditional on macroeconomic variables. However, we also find that its success relies on the forecast accuracy of PD-based credit cycle index.

(F) *A Contribution to macroeconomic stress-testing via migration matrices based on an internal rating system* - Most practices of stress testing focus on PDs (probabilities of default). There are few studies which explicitly link business cycles to rating transitions with applications to stress-testing. Using a one-factor model, we estimate a stressed migration matrix and determine the resulting stressed economic capital.

This thesis presents these contributions in three chapters. The first chapter presents a comprehensive analysis of the internal rating system for a Canadian SME loan portfolio, which is the basis for developing a migration matrix. As above, the SME portfolio contains a considerable default and migration history. In this regard, it provides a laboratory setting to evaluate different methodologies for constructing migration matrices. The second chapter presents three competing methods for constructing migration matrices depending on assumptions concerning the pattern and frequency of observations. Based on the one-factor model, the third chapter develops procedures for forecasting migration matrices with an application to stress testing. Both the second and third chapters present applications of the methodology in the determination of credit risk economic capital.

The more specific contributions of each chapter can be described as follows:

In Chapter 2, we segment data along different dimensions including loan size, risk grade, industry and geography to analyze the architecture of the internal rating system and risk characteristics of the SME loans portfolio. The key observations concerning this portfolio are:

1. *The speculative grade of SME obligors* - In a mapping with an external rating scale using default rates, the *Financing Company* targets high-risk SME obligors that could be classified as speculative grade within the external rating system.

2. *Skewed rating distribution and riskier obligors* - In contrast with the results in the existing literature, the rating distribution of studied SME loan portfolio is skewed to low-risk ratings instead of high-risk ratings. The SME portfolio deals with risker obligors than those investigated in the existing literature.

3. *The concentration features of a SME loan portfolio* - We observe that the *Financing Company*'s portfolio has several concentration features: obligor concentration on small loans, exposure concentration on large loans, regional concentration in Quebec and Ontario, and sector dominance in *Manufacturing*.

In Chapter 3, motivated by the special features of our internal ratings data (incomplete observations and varying time observation intervals), we compare and analyze appropriate estimation methods of migration matrices to accommodate such data under a one-year time horizon. The drivers of differences among the three competing methods under the different patterns of observation data are investigated. The three approaches are applied to annual data of credit rating histories in three sub-samples organized by loan size. The empirical differences of these migration matrices estimates are analyzed using different metrics in the determination of credit VaR. The main contributions in this chapter are:

4. *The first comprehensive comparison among three competing methods of migration estimation* - The difference among the three competing methods are examined from different assumptions concerning the pattern of observation to reveal the bases of their empirical differences. Indeed, they are homogeneous multi-state Markov model (HMM) for different observation patterns of data. The Cohort method assumes equally-spaced and identical observation time data in the context of a discrete time setting. The Duration method is based on continuous time observation data, while the MT-Duration method is for data with a mixed discrete-continuous observation pattern.

5. *A new treatment proposed to deal with NR obligors* - Instead of treating NR obligors as non-informative, we assign NR obligors as a censored-state at the end of period based on the MT-Duration method so that all the possible rating states that could have been visited will contribute to the maximum likelihood function for migration estimation.

6. *Credit risk difference derived from the three competing migration matrices estimates* - we apply the estimates of migration matrices based on the three competing method to a restricted version of the CreditMetric model to assess the credit VaR (Value at Risk) of the SME loan portfolio. The Cohort method tends to overestimates the default risk,

while the Duration underestimate it. Capturing missing information, MT-Duration provides estimates between the two.

7. *Results in examining the assumptions of the Markov property and time homogeneity* - We find evidence for SMEs' rating reversal activity contrary to the rating drift seen in the external rating data. However, the default probabilities of SME obligors are more sensitive to a prior downgrading history. Using the Aalen-Johansen estimator to estimate migration matrices for inhomogeneous Markov chains, we observe only a marginal difference between the continuous-time migration matrices estimates with and without the assumption of time-homogeneity for a one-year time horizon.

In Chapter 4, using both Cohort and Duration migration estimates, we calibrate a one-factor model to examine both in-sample and out-of-sample performance. With the predicted migration matrices, we demonstrate to what degree the migration matrices impact portfolio credit value and potential credit losses. Finally, a hypothetical stress scenario is selected to evaluate the impact on migration matrices and the corresponding stressed credit loss. The key findings in this chapter are:

8. *The construction of a credit cycle index* - Using both Cohort and Duration quarterly migration matrices, we construct the credit cycle index from the default rates reflecting the impact of macroeconomic variables. In contrast to previous results that used external rating data, the significant macroeconomic variables on SME rating process are more related to financial markets instead of standard macroeconomic variables such as the unemployment rate and GDP. In addition, we found evidence that our explanatory variables explain less default behavior for the SME loan portfolio than does the one associated with a large corporate bond portfolio.

9. *Advantages of the one-factor model to forecast and stress migration matrices* - we use a one-factor structural model which, as suggested by Basel II, is compatible with the internal rating system of banks. As a result, both PDs and migration rates can be stressed within one framework. Another advantage of the one-factor model is that it simplifies the model development if the financial institution already has a PD forecast model that incorporates macroeconomic variables.

10. *The importance of the migration matrix for the determination of the accuracy of credit loss assessment and stress testing* - The analysis of corporate credit risk requires not only the specification of default loss but also mark-to-market loss through the migration matrix. We apply one-factor model forecasts to evaluate an artificial portfolio using the CreditMetric model. Compared to the credit loss obtained from migration matrices of two naive approaches, we find that the one-factor model migration matrices reduce the deviation by 4-5 times. The factor model is also used to access the credit loss change of a portfolio under a stressed scenario based on Cohort and Duration estimates.

Chapter 2

The Internal Rating System For a Canadian SME Loans Portfolio

2.1 Introduction

This thesis uses unique and rich internal rating data associated with a Canadian Small-Medium Enterprise (SME) loans portfolio from 1998 to 2009. The data presents a significant contrast with existing literature which either was based on external ratings (i.e. Moody's or Standard&Poor's) or relied on the internal ratings of a wholesale commercial loan portfolio with, in many cases, limited historical span. For example, the first published study on rating migration analysis (Araten et al., 2004) drew upon six years (1997-2002) of internal ratings data from JPMorgan Chase's wholesale exposure for about 33,000 distinct obligors. The data used in Gagliardini and Gourieoux (2005) originated from the Banque de France covering 1992 to 2003. It concerns about 30,000 firms which are in general of small and medium size for the wholesale and retail sectors. Mählmann (2006) was based on the internal rating database of a medium-sized German bank spanning a period from 1996 to 2002. The majority (more than 85%) of borrowers in its portfolio are SMEs. The availability of such unique data in this thesis is an important contribution to the analysis of credit migrations that emphasizes the internal ratings of a SME loan portfolio.

The data analysis is a primary step in the empirical research. Before we proceed to the modeling on credit migration, it is important to study data characteristics for a better understanding of the risk patterns characterizing such a portfolio and to filter the data for consistent quality. This chapter gives a comprehensive description of the internal rating database of a Canadian financing company.

The outline of this chapter is as follows. Section 2 describes the *Financing Company* and the characteristics of Small-Medium Enterprises (SMEs) in its portfolio. Section 3 explores the architecture of the studied internal rating system, including rating scale, rating process, and rating philosophy. Section 4 provides a detailed description of portfolio data and sets up the sample data using various data filters. Finally, in Sections 5 and 6, rating distributions and default rates are analyzed along different dimensions including time, risk grade, industry and geography. The chapter concludes with section 7.

2.2 Financing Company

In contrast to normal commercial financial institutions, the *Financing Company*, whose internal rating system is studied in this thesis, delivers financial services to Canadian small and medium-sized enterprises (SMEs) who have difficulty obtaining commercial loans from traditional sources. Specially, the *Financing Company* pays particular attention to start-ups, innovators, and fast-growing companies.

SMEs are companies whose headcount or turnover falls below certain limits. Industry Canada defines a small business as one that has fewer than 100 employees (if the business is a goods-producing business) or fewer than 50 employees (if the business is a service-based business), and a medium-sized business as fewer than 500. Meanwhile, SMEs must have annual sales revenues between $30,000 and $ 50 million. Approximately 97% of the businesses serviced by the *Financing Company* can be considered small, 2% medium-sized and only 0.3% large.

SMEs are now recognized world-wide to be a key source of dynamism, innovation and flexibility in advanced industrialized countries, as well as in emerging and developing economies. They play an important role in the world economy and contribute substantially to income, output and employment. Estimated data for the 27 countries in the European Union (the EU-27) for 2012 illustrates the importance of SMEs. They account for 99.8% of all enterprises, employ 67% of all workers and contribute 58% of gross value added (GVA) defined as the value of their outputs less the value of intermediate consumption (Edinburgh Group, 2013). In Canada, small businesses account for more than 98% of all firms and proportionally play a large role in net job creation, creating 78% of all new private jobs from 2002 to 2012. SMEs accounted for 52% of private sector GDP and 41 % of Canada's total value of exports in 2011 (Industry Canada, 2013).

However, the presence of financing gaps in the debt market for SMEs has been postulated for some time. In the business loan category, the SME sector is characterised by wider variances of profitability and growth than for larger enterprises. SMEs, for instance, exhibit greater year-to-year volatility in earnings, and the survival rate for SMEs is considerably lower than that for larger firms (OECD, 2006). Thus, the *Financing Company* is positioned to provide a lending solution to a segment of the SMEs loans market that is at least qualitatively riskier than the rest of the market.

2.3 Architecture of the Internal Rating System

Internal credit ratings are becoming increasingly important in credit risk management, especially with the requirements of implementation of an internal rating based (IRB) approach encouraged by Basel II. Risk ratings are the primary summary indicator of risk for a bank's individual credit exposures [1]. Understanding how rating systems are conceptualized, designed, operated, and used in risk management is thus essential to understanding how banks perform their business lending function and how they choose to control risk exposures.

In choosing the architecture of its rating system, a bank must decide which loss concepts (for example, default, recovery, exposure, and expected loss) to employ, the number and meaning of grades on the rating scale corresponding to each loss concept, and whether to include "watch" grades on such scales. Treacy and Carey (1998) found that 60 % of the surveyed 50 largest US banks have one-dimensional rating systems in which ratings approximate expected loss; 40% have two-dimensional systems in which default risk of obligors is appraised by one scale, while expected loss of individual exposure is appraised on another. The *Financing Company* uses a two-dimensional system to separate the assessment of default risk through a risk rating system and recovery risk through a security coverage system. This thesis only has access to the risk rating system in terms of the probability of default (PD), which gives an opinion of the obligor's overall capacity to meet its financial obligations.

Generally, a rating scale can be expressed by letter or number with rating modifiers. The *Financing Company* rating system uses numbers to differentiate the rating grade from 1 to 5.5 in intervals of 0.5 for non-defaulted obligors, with 1 representing the safest category and 5.5 representing the riskiest one. In all, there are total 10 grades. Previously defaulted or impaired obligors who have been cured are classified into a "watch-list". They are excluded from our analysis.

In an external rating system with letter rating scales, ratings in the four highest categories (AAA, AA, A, and BBB) generally are recognized as being investment grade, whereas debt rated BB or below generally are regarded as non-investment grade or speculative grade. In a mapping with an external rating scale using default rates (see Table 2.1), the highest internal rating category of *Financing Company* generally corresponds to the B broad rating class of an external rating system. Therefore, the *Financing Company* targets high-risk SME obligors that could be classified as speculative grade within the external rating system.

The rating process of the *Financing Company*'s internal rating system was not disclosed

[1] We follow industry usage by referring to individual loans or commitments as "facilities" and overall credit risk arising from such transactions as "exposure".

in detail for our empirical analysis. Basically, the *Financing Company* adopts two scoring models for obligors on either side of the \$ 250,000 size threshold. A common risk rating is assigned with the reconciliation of scoring models. This methodology is used to account for the differences in available financial information for obligors on either side of the threshold. Risk ratings are assigned at authorization and are reviewed periodically. Generally, the ratings are monitored at least once a year for an external rating system. The *Financing Company*, like many financial institutions, is faced with a different situation. The expenditure of resources at a rate similar to that of the rating agencies would compromise profitability. In addition, monitoring SMEs with poor and infrequent information is much more costly. From what we know, the *Financing Company* reviews risk rating at one-year intervals only for obligors with loan size above \$1,000,000.

Recently, there has been some literature dealing with the philosophy and classification of different types of rating systems. An important classification of rating systems is the decision whether a rating system is point-in-time (PIT) or through-the-cycle (TTC). Loosely speaking, the PIT approach can be thought of as using all available and pertinent information when computing the default risk metrics that are mapped into ratings. On the other hand,the TTC approach primarily reflects long-run, enduring credit risk trends which is supposed to balance the need for accurate default estimates and the desire to achieve rating stability. Compared with PIT rating systems, TTC ratings display much less volatility and migrations over cycles. These two types have to be considered as extreme types of possible rating methodologies. External rating agencies normally adopted TTC ratings system. Most internal rating systems are somewhere between these two methods and are neither PIT or TTC in a pure fashion. In the last chapter, the quantified cyclical behavior of rating migration provides evidence to suggest that the rating philosophy of *Financing Company*'s internal rating system is PIT.

2.4 Portfolio Data

The *Financing Company* loans portfolio analyzed in this thesis covers the time period from April 1, 1998 to March 31, 2010, which provides annual and quarterly migration matrices from the year 1998 to the year 2009. In the *Financing Company*'s database, the fiscal year starts from calendar month 4 and the annual rating is the rating of calendar month 3. Accordingly, we use ratings of calendar months 3, 6, 9, 12 for quarterly ratings. Therefore, the annual migration matrices covers the time period between months 3 of two adjacent calendar years. For example, the 1998 migration matrix refers to the time period from April 1, 1998 to March 31, 1999. For each obligor in the *Financing Company* loans portfolio, the data consist of a risk rating, dollars outstanding, industry, and geography at a given time. The loan size associated with an obligor is defined as the maximum dollar outstanding in its history.

To ensure the data analyzed in this thesis is of high quality, we apply various filters. First, a size cutoff of \$50k in loan size is imposed in order to avoid noise. As shown in Table 2.2, the loan size group (0, \$50k) comprises approximately 11% of the portfolio's obligors but only 0.7% of its exposure. Accordingly, we will investigate 4 sample groups: subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and the total sample with loan size [\$50k, $+\infty$). Second, the obligor observation is at the parent level to avoid multiple counting of distinct subsidiaries with characteristics similar to the parents. Third, a simpler 7-credit position rating system (rather than the 10 described in Section 2.3) is built up to avoid statistical bias caused by small sample sizes. We observe in Table 2.2 that rating 5.5 appears after 2006Q4 with only 0.17% of total sample records. Meanwhile, the observations for rating 5 for subsamples B and C consist of only 1.99% and 1.04% of subsample records respectively. Furthermore, rating 1 has a default rate similar to that of rating 1.5 (see Table 2.1), consisting of a small proportion (3.53%) of total sample observations. Therefore, we consider a rating 1-1.5 as 1.5 and a rating 4.5-5.5 as 4.5. This methodology reduces the rating system from 10 to 7 rating categories (1.5, 2, 2.5, 3, 3.5, 4, 4.5) excluding default (D). Fourth, we exclude new credit exposures that arrive in the middle of the studied period in the analysis.

It is worth noting that some observations have no ending state in the studied period; we assign "Non-rated"(NR) rating to these observations. NR ratings are observed when obligors have a rating at the beginning of a studied period but do not have rating or any exposure at the end of the studied period: they are firms that do not need to borrow or that roll over their debt with another lender. But these are different from the case of "Withdrawn" (WR) in rating agencies where an external agency may decline to (or are requested by a company not to) publish a rating.

The final database studied in this thesis contains 202,019 years observations based on 15,095 defaulted and 45,087 non-defaulted obligors. Unless noted otherwise, this thesis focuses on obligor-year observations. Examining the distribution of exposure and obligors by loan size in Table 2.3, we observe a monotonic decrease of obligor numbers with size (50%, 37% and 14% for subsample A, B, C respectively) and a sharp increase of exposure amount with size (11%, 33% and 56% for three subsample groups). In an extensive examination of bank balance sheets, Carey (2001) found that the largest 10% of borrowers generally account for approximately 40% of total exposure in banks' commercial loan portfolios. Our subsample group C with size bucket [\$1000k, $+\infty$) comprises 14% of the obligors but holds 56% exposures, indicating a heavy concentration of exposure among a relatively small proportion of borrowers in the *Financing Company*'s portfolio.

2.5 Rating Distributions

Different rating systems produce specific rating distributions that reflect the nature of the portfolio. Accordingly, in the following subsections, rating distribution will be investigated along different dimensions including loan size, sector, geography and cross-sectional dimensions.

2.5.1 Rating Distributions in General

First, we begin with the overall rating distribution. In some credit risk models, the rating distribution is assumed to be normal for simplicity, an assumption that does not to correspond to actual ratings. In the existing literature, rating distributions are more skewed to high-credit quality grades. For example, Bangia et al. (2002) provide the average ratings distribution based on the Standard & Poor's $CreditPro^{TM}$ 3.0 database from 1981 to 1998. Over 7 rating categories[2], the distribution was skewed to the third rating category A. Investment rated (above rating BBB) obligors were 72% of the dataset. In analyzing the rating distribution of an internal rating system from one German bank by firm size, Mählmann (2006) found that 30% of obligor ratings were in rating grade 2 out of 6 rating categories. They were skewed to the high-level ratings as well.

The *Financing Company* portfolio of SME loans has different features as seen in Table 2.3 and Figure 2.1:

1. The rating distribution of our SME loan portfolio is skewed to low risk ratings. Approximately 43% obligors are rated in risk ratings 4 and 4.5.

2. The rating distribution of Subsample A is highly skewed to the low rating category. Almost half of the obligors (52%) in subsample A are rated in the lowest rating grades 4 and 4.5.

3. The rating distribution of subsample B is less skewed to the low rating category. Some 37% of all olbigors are located in ratings 4-4.5. This relative decrease in skewness results from fewer obligors in risk rating 4.5 (18% in subsample B vs. 33% in subsample A). Meanwhile, there are more obligors in subsample B having rating 1.5-2 (13% in rating 1.5 and 10% in rating 2).

[2] The S&P's letter rating scale comprises 17 different rating categories as well as default and withdrawn states. The author reduces them to 7 rating categories exclude rating modifiers.

4. The rating distribution of Subsample C is relatively bell shaped except at the point of highest rating 1.5 which composes 20% of all obligors.

To summarize: in contrast to results in the existing literature, we observe skewed rating distribution of the SME loan portfolio towards ratings of lower credit quality. This result is reinforced by the finding in Section 2.3 that SME obligors of the *Financing Company* are equivalent to speculative grade in external rating systems. In addition, we found that this skewed-rating distribution phenomenon as well as the riskiness of obligors decrease monotonically with loan size so that subsample A is the most likely to have the highest proportion of obligors in the 4 and 4.5 risk ratings, while subsample C has the least.

2.5.2 Rating Distributions Over Time

The rating data spans 12 years from 1998 to 2009. Figure 2.2 exhibits the evolution of rating distributions over time. Table 2.4 and Figure 2.3 provide a breakdown across loan size and time. The main findings revealed by the table and figures are summarized below.

1. In general, the number of active obligors increased from year 1998 to 2008 but declined slightly in 2009 as shown in Figure 2.2. There was a 100% increase of obligors over 12 years. The increasing trend coincides with the global boom of the financial sector but ceased during the financial crisis 2008-2009.

2. Examining the evolution of rating distribution for subsample A with size [\$50k, \$250k) in Table 2.4, we observe first that the active obligors number in high credit quality ratings 1.5, 2 and 2.5 monotonically increase over time. Second, we see that the percentage of obligors in two extreme rating statuses (1.5 - 2 and 4.5) more than doubled over the period. Taking rating 1.5 and 4.5 as example, the percentage of obligors for rating 1.5 increased from 6% in 1998 to 12% in 2009, and the percentage of obligors in rating 4.5 rose from 15% in 1998 to 35% in 2009. These movements are offset by decreasing percentages for the middle ratings.

3. Subsample B shared similar features with A. The low-risk ratings increased, particularly 1.5 which more than doubled its share. The middle ratings' share decreased significantly, while rating 4.5 exhibited decided share growth from 1998 to 2002, tapering off somewhat at the end of the period.

4. We see a similar pattern in subsample C. The shares of ratings 3, 3.5, 4 decreased significantly while the share of ratings 1.5 and 4.5 increased. Accordingly, the highest

share of ratings changed from rating 3 with 21% in 1998 to rating 1.5 with 25% in 2009.

In sum, the *Financing Company* portfolio has overtime increased the share of obligors in extreme ratings. Meanwhile, the best rating 1.5 is more related to obligors with larger loan size and the worst rating 4.5 is more related to those with smaller size. Accordingly, we observe in Figure 2.3 that the rating distribution curve of the total sample evolved over twelve years to more skewed-right bell shape in subsample A, to flat-top bell shape in subsample B, and to skew-left bell shape in subsample C.

2.5.3 Rating Distributions By Sector

The *Financing Company* lending activities encompass various sectors of the Canadian economy. The obligors are classified into 11 industries: *Business Services*, *Construction*, *Manufacturing*, *Non-Business Services*, *Resources*, *Retail*, *Supplier of Premises*, *Tourism*, *Transportation & Storage*, *Wholesale* and *Other*. Figure 2.4 and Table 2.5 exhibit sector composition of obligors. Table 2.6 and Figure 2.5 describes the rating distribution across sectors and loan size. Results from these figures and tables can be described as follows:

1. *Manufacturing* is the industry which has the largest number of active obligors, containing over 26% of all obligor years in the portfolio. Next are *Retail*, *Tourism* and *Wholesale* with proportions of 12.4%, 11.6% and 10.4% respectively.

2. We see that the predominance of *Manufacturing* carries through all subsamples with different loan size. Within *Manufacturing*, the proportion of Manufacturing obligors are 22.2%, 27.5% and 39.2% in subsample A, B and C. Figure 2.4 clearly represents this increasing patten with size in *Manufacturing*. While in other industries, such as *Business Services*, *Non-Business Services*, *Construction* and *Retail*, we observe the opposite, decreasing patten.

3. The top panel of Table 2.6 exhibits the rating distribution for each industrial sector. We observe a skewed distribution to riskier ratings in the industrial sectors, especially *Supplier of Premises* and *Tourism* with around 63%, 55% obligors rated in the high risk ratings (4 and 4.5) respectively. The risk ratings 4 and 4.5 have the largest proportion of obligor years in the portfolio.

4. Subsample A has a highly skewed rating distribution towards high-risk grades. The sectors with most skewed distribution of ratings are still *Supplier of Premises* and

Tourism with 72% and 63% ratings in high risk grades (4 and 4.5). Even *Manufacturing* has 30% proportion of obligors rated in the worst rating grade 4.5, which is much higher than the 18% for *Manufacturing* total sample.

5. The sectoral rating distribution in subsample B is much less skewed compared with that of subsample A. Except for *Supplier of Premises* and *Tourism*, the proportions of obligors rated in rating 4.5 for the other industrial sectors sharply decrease to 20% or less, while the proportion for *Manufacturing* goes to only 13%.

6. Compared with other subsample groups, subsample C has bell-shaped curve rating distribution for each sector. In *Business Services*, *Construction* and *Manufacturing*, the propositions of rating 1.5 and 2 constitute 35%, 37% and 38% respectively. We notice that *Supplier of Premises* and *Tourism* still have a skewed rating distributions.

In summary, we observe that *Manufacturing* is the largest in terms of obligor years, accounting for over a quarter of the portfolio. This predominance is a feature of all subsamples. Meanwhile, the proportion of *Manufacturing* increases with loan size. In subsample C, less than half obligors are in *Manufacturing*. A skewed-shape distribution towards high-risk grades is exhibited for most of the sectors in the portfolio. However, the skewness decreases with loan size. In subsample C, we observe the bell-shaped curve for most industries except *Supplier of Premises* and *Tourism*.

2.5.4 Rating Distributions By Geography

Financing Company provides SME financing services across twelve Canadian regions: Newfoundland & Labrador (N. & L.); Prince Edward Island (P.E.I.); Nova Scotia (N.S.); New Brunswick (N.B.); Quebec (QC); Ontario (ON); Manitoba (MN), Saskatchewan (SK), Alberta (AL); British Columbia (B.C.); the Yukon (YK), Northwest Territories and Nunavut (N.W.T.). Figure 2.6 exhibits the geography composition of sample groups and Table 2.7 describes the rating distribution of sample groups by geographical regions. Results are summarized as follows.

1. Figure 2.6 demonstrates the obvious geographical concentration of obligors. The dominant regions of SME loan portfolio are Quebec and Ontario, accounting for 34% and 30% respectively. The regions of Northwest Territories and Nunavut, Prince Edward Island and Yukon consist of only 1.4% of obligors in total.

2. Examining the geographical regions composition in different sample groups, we observe that Quebec proportions increase with loan size. The proportions in Quebec for subsample A, B and C are respectively 32%, 35% and 39%, shown in Table 2.7.

3. For most regions, the rating distributions in the total sample group are skewed to the high-risk level ratings as shown in Figure 2.7. The most skewed curve comes from Prince Edward Island with 37% of obligors rated in rating 4.5 and 24% rated in rating 4. Interestingly, we observe the distribution of ratings in Quebec province is the least skewed with 17% and 18% olibors rated in rating 4 and 4.5 respectively.

4. The regional rating distribution in subsample A is heavily skewed to riskier ratings, especially Prince Edward Island, with 69% obligors located in ratings 4 and 4.5. For the two largest regions, Quebec and Ontario, we found the proportion of obligors in rating 4.5 increase sharply to 26% and 38% compared with those in total sample, 18% and 28% respectively.

5. In subsample B, the regional rating distribution is less skewed to high risk level ratings. All regions except Prince Edward Island start to have more olibgors in the high-credit quality ratings.

6. Except Prince Edward Island and Nova Scotia, the rating distribution of most regions for subsample C are bimodal with one mode at rating 1.5 and the other mode at rating 3. Consider Ontario and Quebec for example: the proportions of obligors at these two modes are around 20% and 18% respectively for the two regions.

7. Comparing the three different loan size sample groups, we observe that the regional proportions of obligors in rating 1.5 monotonically increases with growth of loan size from around 8% in subsample A to 20% in subsample B; while those proportions in rating 4.5 decreased significantly from around 40% to around 10% respectively.

To summarize, nearly two third of obligors of the *Financing Comapny* are concentrated in the provinces of Ontario and Quebec. This predominance is reinforced with loan size. For example, in subsample C with large loans above \$1000k, Quebec contains almost 40% of obligors. The rating distributions of geographical regions in the total sample are skewed to the high-risk ratings, especially for Prince Edward Island with over 60% obligors in high risk ratings 4 and 4.5, compared with only 36% for Quebec. The effect of loan size on the rating distribution shape observed in previous sections is demonstrated again in the regional analysis. Larger loans are inclined to have better credit quality ratings.

2.6 Default Rate

According to Basel II, a default is defined if either or both of the following events occurs: (i) the bank determines that the borrower is unlikely to pay its obligations to the bank in full, without recourse to actions by the bank such as the realization of collateral; or (ii) the borrower is more than 90 days past due on principal or interest on any material obligation to the bank. As the last component of credit quality assessment, default events are usually examined through historical default rates. The default rate is a critical piece of information in determining the portfolio loss distribution. In the later part of the thesis, we will investigate different estimation methodologies and a forecasting model for the default rate in the context of a migration matrix. In this section, we will examine characteristics of default rate along the dimensions introduced in the preceding sections.

2.6.1 Default Rates By Rating

Table 2.9 and Figure 2.8 exhibit the weighted-average one-year default rates of sample data segmented by risk ratings over the period from 1998 to 2009. The weighted-average one-year default rate, weighted by obligor years, is equivalent to adding up all the annual defaults in a given period divided by the sum of all the annual obligor years in that period. The findings are summarized below.

1. Figure 2.8 shows monotonically increasing default rates with rating. The average default rate is 7.6%. Specifically, in the total sample group, we observe a default rate of 2.9% at rating 1.5 and that of 11.6% at rating 4.5. This negative correlation between credit quality and default rate occurs in all three subsample groups.

2. We see that the increasing pace of default rates among the ratings are not steady. In total sample group, the default rates smoothly rise up from rating 1.5 to 4 with around 1% per category change, but jump around 3% from rating 4 to 4.5. This feature is also observed in all three subsample groups.

3. Subsample B has lower default rates than C for all ratings. Subsample A has the highest default rates among three subsample groups except for the middle ratings of 3.5 and 4, where we observe increasing default rates with loan size.

In summary, default rates demonstrate monotonic increase with the deterioration of risk rating with an average default rate of 7.6%. Comparing the three subsample groups, we observe the highest default rates in subsamples A and C. Subsample A contains the highest default rates in low-risk and high-risk ratings, while subsample C has the highest default rates in the middle ratings (3.5 and 4).

2.6.2 Default Rate Over Time

Table 2.10 exhibits the time series of annual default rates from 1998 to 2009. Figure 2.9 shows the relationship between loan size and default rate. The results are described as follows.

1. In the total sample, there are three high points of default rates over time: 8.3% in year 1999, 7.5% in year 2002 and 9.4% in 2008. If we investigate economic events during the years 1998-2009, there was the Russian financial crisis in 1998, the Brazil crisis in 1999, the crash of the dot-com bubble in 2000-2001, and the financial crisis in 2007-2009. This reflects the influence of the business cycle on default rates.

2. Comparing the three subsample groups, Figure 2.9 indicates that subsample A with small loans has the highest annual default rates among the three subsample groups. Between subsamples B and C, there is no pattern of absolute dominance over time except during the period of 1998 - 2000 when subsample C has very low default rates 4.6-6.7%.

3. The differences between default rates among sample groups become gradually larger over the time. For example, the gap between subsample A and B increases from 0.2% in 2001 to 3.4% in 2008, while the discrepancy between subsample A and C raises from 0.3% in 2002 to 3.4% in 2009. This suggests that small obligors have become riskier over time.

4. The minimum and maximum default rates in subsamples A and B occur in 2006 and 2008, while those points in subsample C are in 2007 and 2009. This feature maybe due to the fact that small loans are more immediately sensitive to the market situation than the large loans.

In sum, the time series of default rates present the counter-cyclical default behavior of SME portfolio. Default rates raise in adverse macroeconomic environments and decline in good periods. Specifically, we observe a spike during financial crisis period of 2008-2009 in all sample groups. Looking within the subsample groups, we notice that the default behaviors of large loans are less sensitive to business cycle than small loans. As well, we observe that small loans are riskier than larger ones with the fact of increasing discrepancy of annual default rates between subsample A and the other two subsample groups over time.

2.6.3 Default Rate By Sector

The weighted-average default rates of sample groups by industrial sector for the studied period 1998 to 2009 are shown in Table 2.11 and Figure 2.10. Figure 2.11 examines the industrial default rate with the industrial composition in different subsample groups. We summarize the results as below.

1. The maximum and minimum default rates in the total sample group are in *Resources* and *Supplier of Premises* with default rates of 10.2% and 4.7% respectively.

2. Figure 2.10 exhibits the comparison of default rates among different sample groups. We observe that default rates decrease with loan size in most sectors. In *Tourism*, the default rate in the large loan subsample group C (9.2%) is higher than the other two subsample groups (around 8%).

3. In the total sample, as shown in Figure 2.11, although the three smallest sectors *Resource*, *Others* and *Transport & Storage* only consist of 2.8%, 2.6% and 4% obligors, default rates in these three industries are quite high at 10.2%, 8.1%, and 8.3% respectively. This feature is also observed in the three subsamples.

4. In *Manufacturing*, which is the largest industry in terms of population in all sample groups, we observe the second highest default rate (9.5%) in the total sample groups. With increase loan size, fewer default events are seen in *Manufacturing*.

To summarize, *Resources* and *Manufacturing* are the two riskiest industrial sectors for all sample groups. The negative relation between loan size and the default rate is observed in all segments except Tourism. Considering the sector composition in the portfolio, we found that the small sectors in terms of population have high default rates. For example, *Resource* consists of 2.8% of obligors years but has the maximum default rate of 10.2%. Investigating *Manufacturing* which holds the largest proportion of obligors in the portfolio, we observe that decreasing default rates but the increasing proportion of obligor years with loan size (see Figure 2.11).

2.6.4 Default Rate By Geography

The weighted-average default rates of the three sample groups in different geographical regions are shown in Table 2.12 and Figure 2.12.

1. In the total sample, the maximum default rate of 8.2% occurs both in Nova Scotia and Yukon, while the minimum default rate is found in Northwest Territories at 2.8%. Except regions of Northwest Territories and Saskatchewan, regional default rates range closely between 6% - 8%.

2. Investigating different loan size groups, we observe that Northwest Territories is always the region with the lowest default rates in the three subsample groups. The regions where the highest default rate occurs in subsample A, B and C are Nova Scotia (9.5%), P.E.I.(8.3%) and Yukon (12.2%) respectively.

3. As found in previous sections, small loans continue to present riskier behavior than large loans. Except for New Brunswick, P.E.I. and Yukon, subsample A has higher default rates than the other two subsamples as shown in Figure 2.12.

4. In the two largest regions, Ontario and Quebec, we observe similar default rates for the total sample, 7.8% and 7.9% respectively. Meanwhile, the pattern of decreasing default rates with loan size is presented in these two regions.

In sum, Nova Scotia and Yukon are risker regions than others with the highest default rate of 8.2%. The negative relation between loan size and default rate in a particular region dose not universally hold. We observe it in Alberta, Manitoba, Newfoundland & Labrador, Ontario and Quebec. Comparatively, subsample A exhibits higher default rates than the other two subsamples in most regions .

2.7 Conclusions

In this chapter, we have analyzed the architecture of the internal rating system and risk characteristics of the loan portfolio associated with the *Financing Company*, an institution that makes commercial loans to Canadian small and medium-sized enterprises (SMEs). The portfolio was analyzed along several dimensions including risk rating, loan size, industrial sector, geographical region and time. The distinguishing features of *Financing Company* loan portfolio include the following major points.

1. *The speculative grade of SME obligors* - In a mapping with an external rating scale using default rates, the *Financing Company* targets high-risk SME obligors that could be classified as a speculative grade within the external rating system.

2. *Obligor concentration on small loans* - the *Financing Company* portfolio consists of large number of obligors with small loans, with 50% of obligors having loans less than $250k and 37% of obligors categorized into loan size [$250k, $1000k) (See Table 2.3).

3. *Exposure concentration on large loans* - Although the *Financing Company* contains a large number of small loans, most exposures lie in the small number of large loan obligors. We observe that the largest 14% of obligors, who owning bank above $1000k, account for 56% of total exposures in the SME loan portfolio (See Table 2.3).

4. *Skewed rating distribution and riskier obligors* - Contrasting to the results in the existing literature, the rating distribution of studied SME loan portfolio is skewed to high risk ratings in stead of low risk ratings, with around 43% of obligors in ratings 4 and 4.5 (see Figure 2.1). This skewed patten decreases with loan size (see Figure 2.3). The SME portfolio deals with riskier obligors than those investigated in the existing literature.

5. *More ratings allocated to two extreme ratings over time* - In 1998, the *Financing Company* had 8% of obligors in the lowest risk rating 1.5 and 12% of obligors in highest risk rating 4.5; while in 2009, those proportions doubled to 16% and 24% respectively (See Table 2.4).

6. *Dominant Manufacturing sector in SME loan portfolio* - Among 11 sectors, *Manufacturing* contains approximately a quarter of obligor years in the portfolio (see Table 2.6). This dominance of *Manufacturing* prevails in all different loan size groups.

7. *Regional concentration in Quebec and Ontario* - Nearly two thirds of the obligors are concentrated in Ontario and Quebec. With the growth of loan size, Quebec and Ontario contain more proportion of obligors. As shown in Table 2.7, there are approximately 62%, 64%, 70% of obligors located in these two regions in subsamples A, B and C respectively.

8. *Negative relation between risk ratings and loan size* - In the analysis of rating distribution along dimensions of risk rating, time, sector and geography, we observe always the fact that small loans have more high-risk ratings and large loans have more low-risk ratings. Accordingly, the skewed-right bell shape curve of rating distribution tends to weaken with loan size.

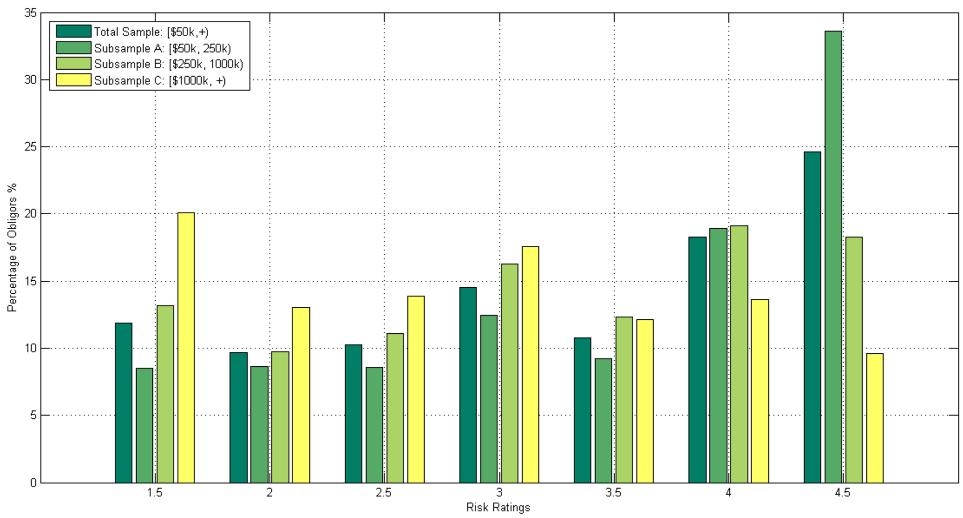
9. *Increasing default events with risk rating* - Consistent with default risk scoring system, the default rates of the sample groups monotonically increase with the deterioration of credit quality. In the total sample group, the default rates among risk ratings range from 2.9% to 11.6% (see Table 2.9), with an average default rate of 7.6%.

10. *Default rates are counter-cyclical* - In the periods of macroeconomic stress, we observe elevated annual default rates in all sample groups (see Figure 2.9). Large loans are less sensitive to the business cycle than small loans.

11. *Small obligors are riskier than large ones* - We note that obligors with small loans (subsample A) have a higher default rate than those with large loans. We also observe that the annual default rates of small loans are significantly higher than those of large loans with increasing differences over time. In the analysis along the dimensions of sector and geography, the riskier behavior of small loans is found in most segments as well. However, it is worth noting that large loans cannot be considered as safer as a rule. We observe increasing default rates with loan size in risk rating 3.5 and 4 as shown in Figure 2.8.

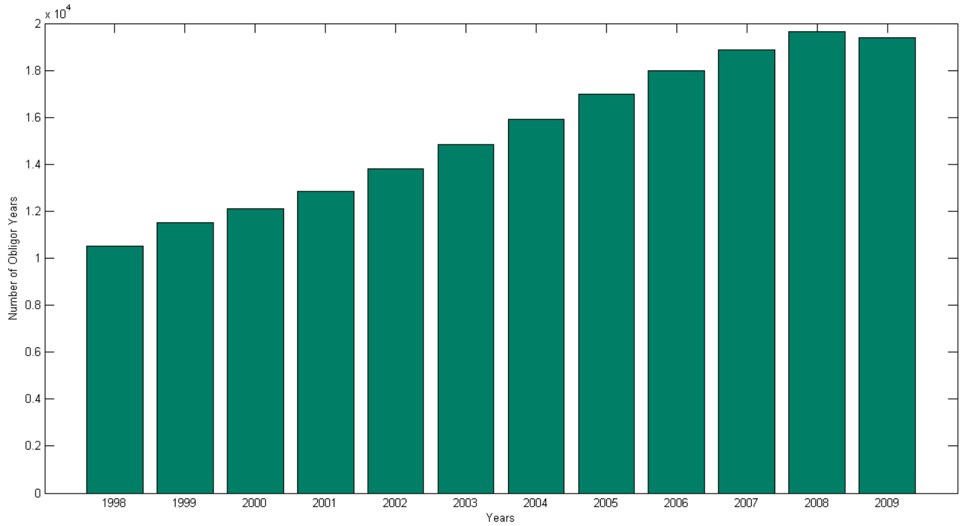
12. *Riskiest Sectors and Regions* - In the segmentation by sector, *Resource* and *Manufacturing* are the two riskiest industries in all sample groups with total sample default rates of 10.2% and 9.5% respectively (see Table 2.11). Nova Scotia and Yukon have the highest default rates of 8.2% in the total sample (see Table 2.12).

Figure 2.1: Rating Distributions of Sample Groups



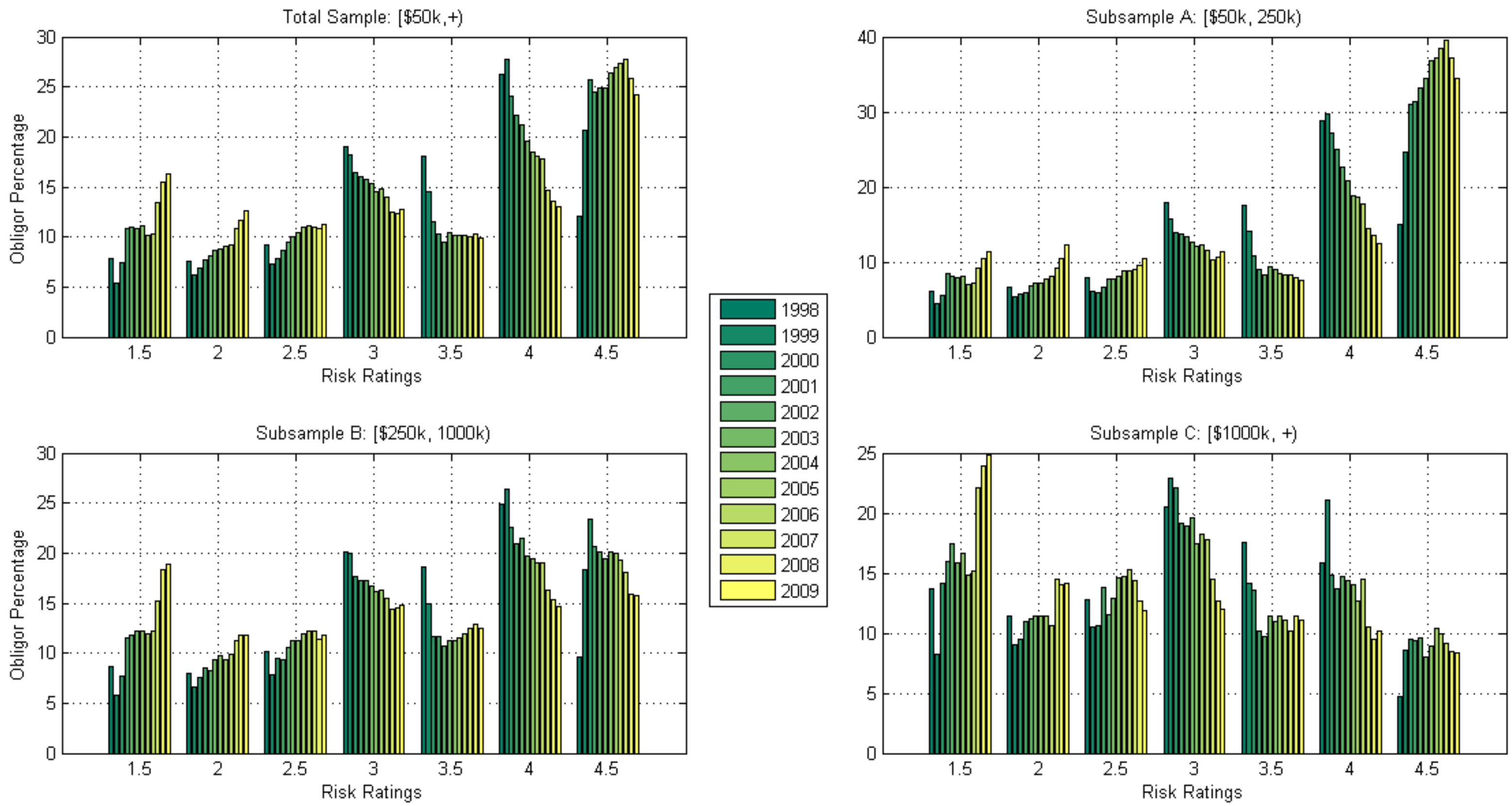
The figure provides the rating distributions of the *Financing Company* portfolio in four sample groups, including subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and total sample. The data is based on obligor-year observations from 1998 to 2009. See Table 2.3 for details.

Figure 2.2: Obligors' Growth Over Time



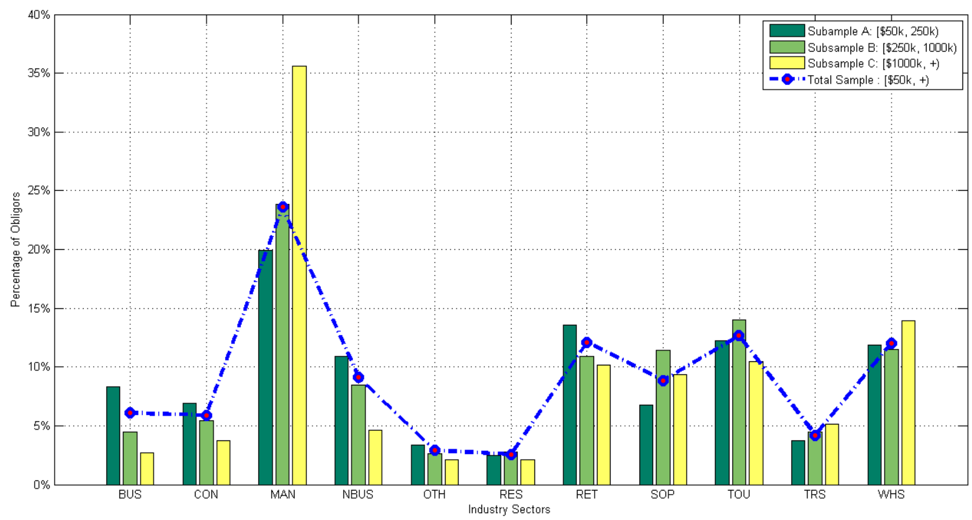
The figure exhibits the evolution of the *Financing Company* portfolio over time. The data gives obligor-year observed from 1998 to 2009.

Figure 2.3: Rating Distributions of Sample Groups Over Time



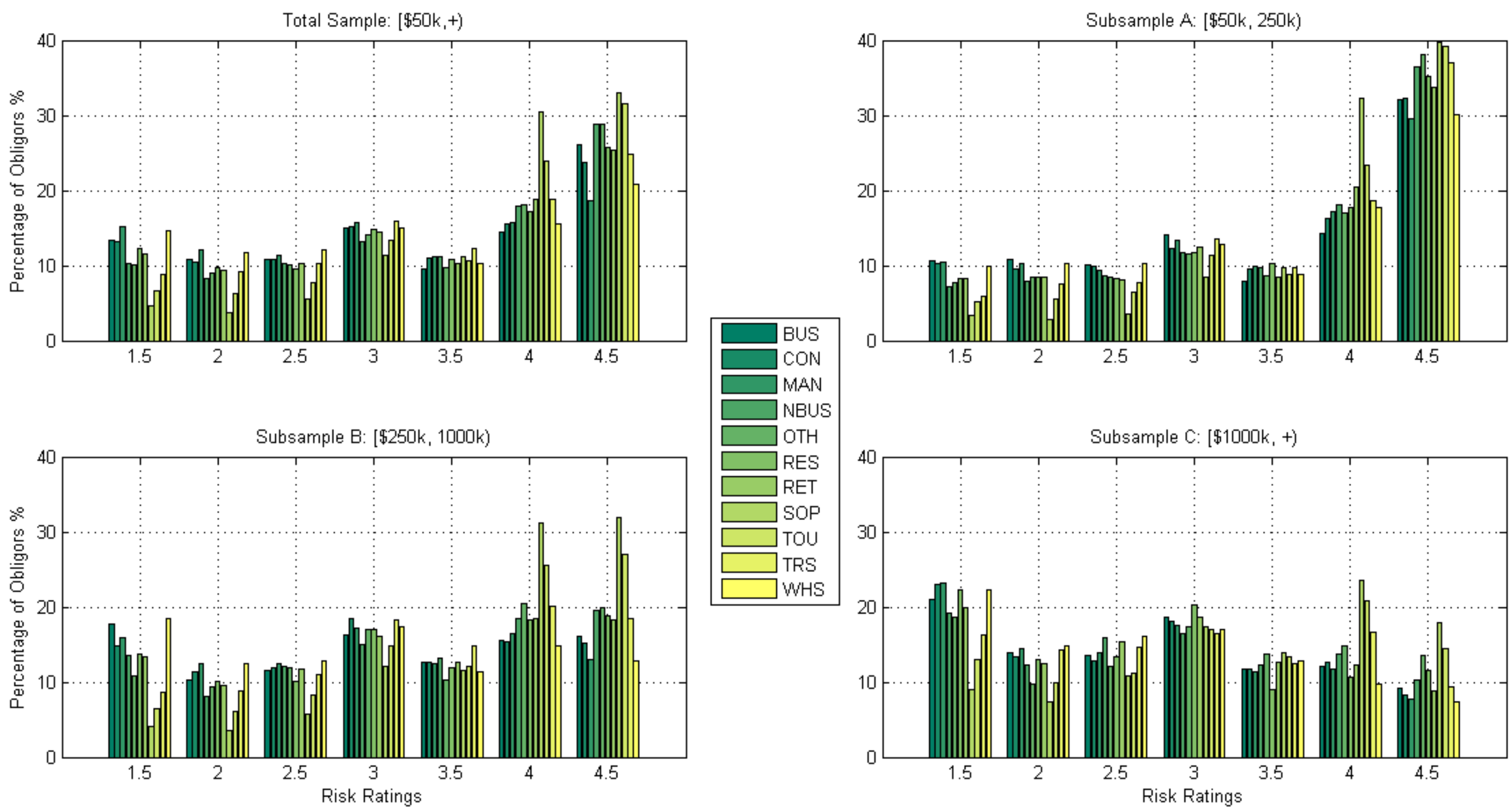
The series of figures shows the rating distributions evolution of the *Financing Company* portfolio over time. The data is based on obligor- year observations from 1998 to 2009. The portfolio consists of obligors rated from 1.5 (least risk) to 4.5 (riskiest) with 0.5 rating intervals. The table is segregated into four panels for total sample, subsample A with loan size [$50k, $250k), subsample B with loan size [$250k, $1000k), and subsample C with loan size [$1000k, $+\infty$).

Figure 2.4: Sector Composition of Sample Groups



The figure describes the sector distribution of the *Financing Company* portfolio in four sample groups, including total sample, subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), and subsample C with loan size [\$1000k, $+\infty$). The data is based on obligor-year observations from 1998 to 2009. The industrial sectors, in alphabetical order, are as follows: Business Services (BUS); Construction (CON); Manufacturing (MAN); Non-Business Services (NBUS); Other (OTH); Resources (RES); Retail (RET); Supplier or Premises (SOP); Tourism (TOU); Transportation and Storage (TRS); Wholesale (WHS).

Figure 2.5: Rating Distributions of Sample Groups by Sector



The series of figures 2.5 describes the sector distribution of the *Financing Company* portfolio in four sample groups, including total sample, subsample A with loan size [$50k, $250k), subsample B with loan size [$250k, $1000k), and subsample C with loan size [$1000k, $+\infty$). The data is based on obligor-year observations from 1998 to 2009. The industrial sectors, in alphabetical order, are as follows: Business Services (BUS); Construction (CON); Manufacturing (MAN); Non-Business Services (NBUS); Other (OTH); Resources (RES); Retail (RET); Supplier or Premises (SOP); Tourism (TOU); Transportation and Storage (TRS); Wholesale (WHS).

Figure 2.6: Geographical Composition of Sample Groups

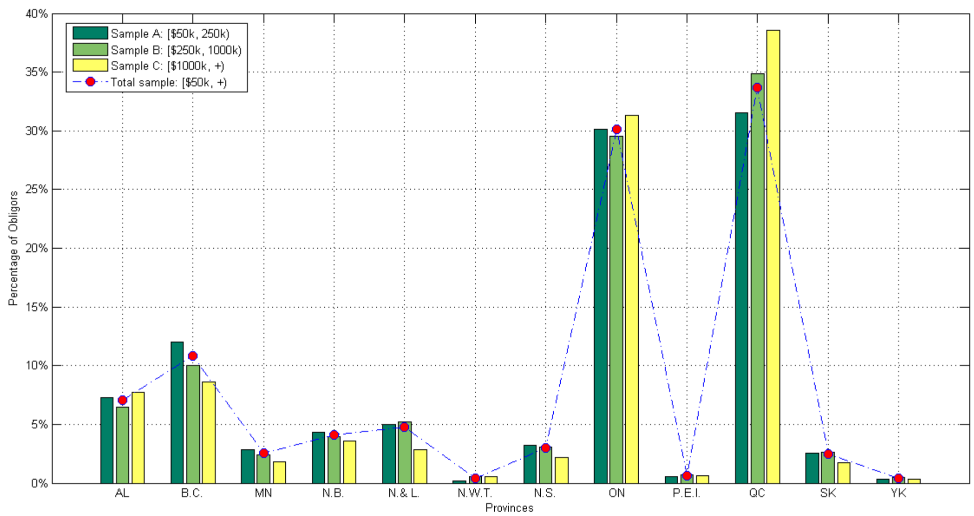
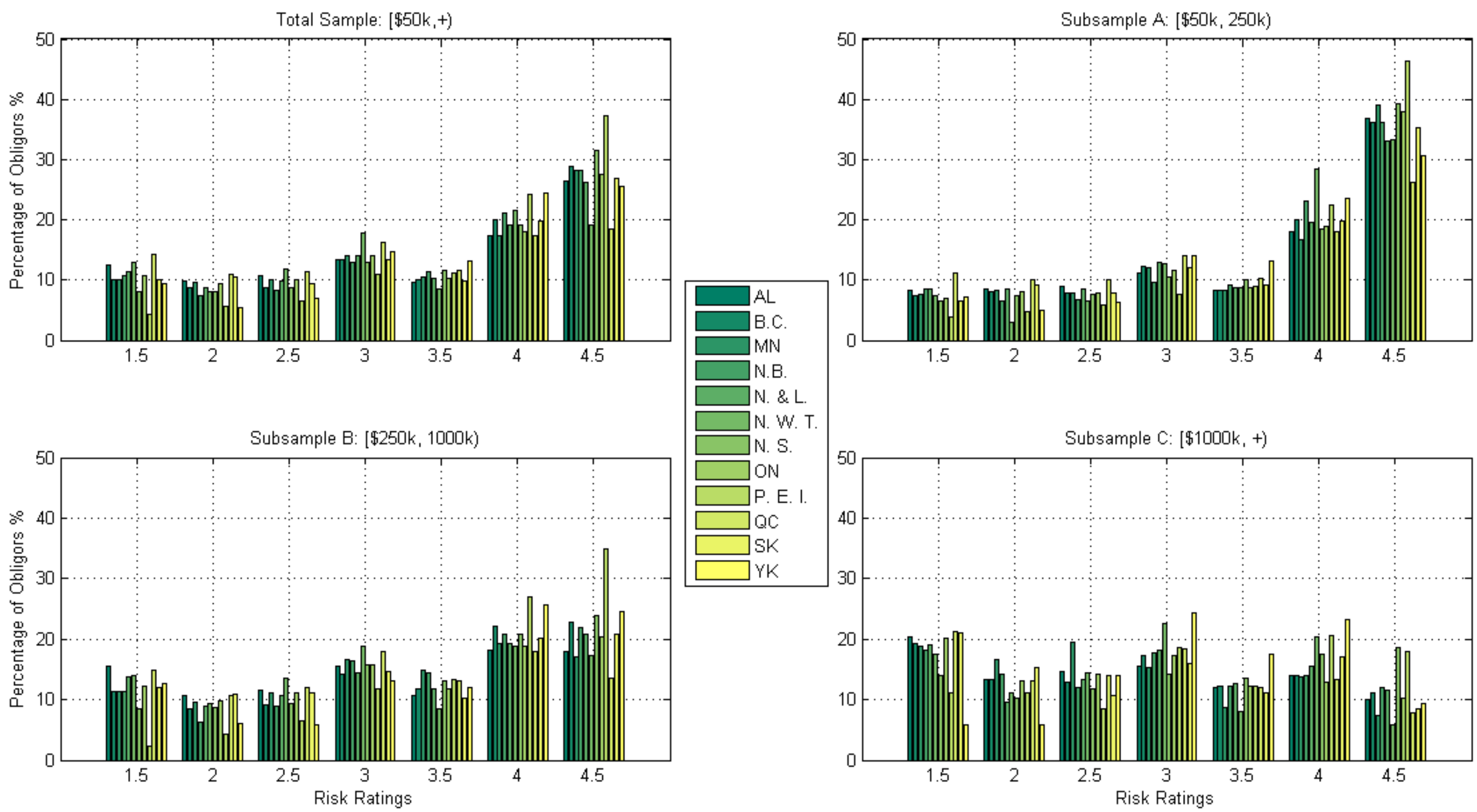


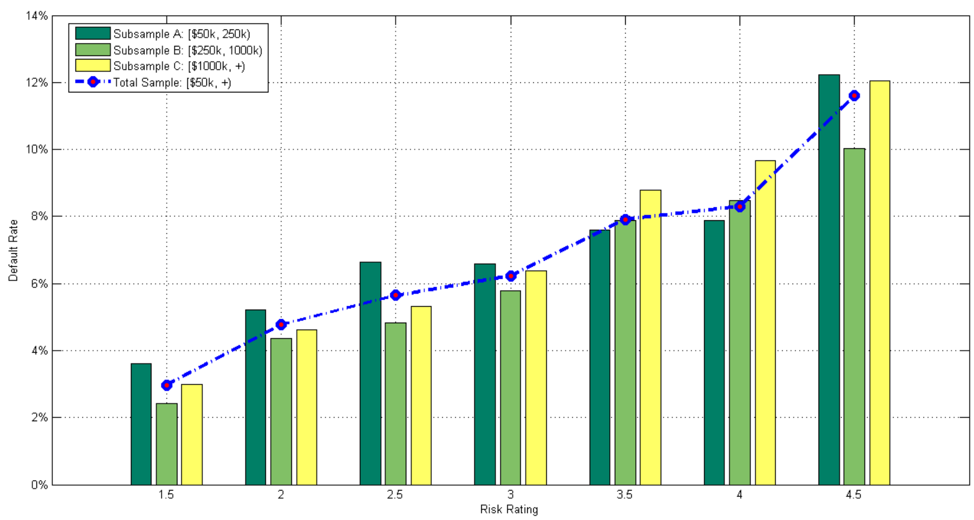
Figure 2.6 provides geographical composition of the *Financing Company* portfolio in four sample groups, including total sample, subsample A with loan size [$50k, $250k), subsample B with loan size [$250k, $1000k), and subsample C with loan size [$1000k, $+\infty$) and the total sample. The data is based on obligor-year observations from 1998 to 2009. The regions include Newfoundland & Labrador (N. & L.), Prince Edward Island (P.E.I.), Nova Scotia (N.S.), New Brunswick (N.B.), Quebec (QC), Ontario (ON), Manitoba (MN), Saskatchewan (SK), Alberta (AL), British Columbia (B.C.), the Yukon (YK), Northwest Territories and Nunavut (N.W.T.).

Figure 2.7: Rating Distributions of Sample Groups by Geography



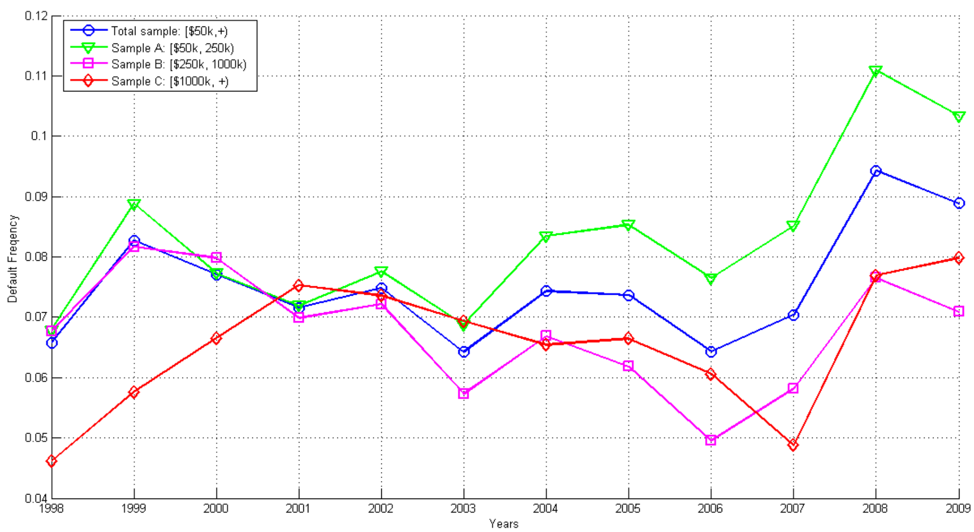
The series of figures provides rating distributions of portfolio by geographical regions in different sample groups. The data is based on obligor-year observations from 1998 to 2009. The regions, in alphabetical order, are as follows: Newfoundland & Labrador (N. & L.), Prince Edward Island (P.E.I.), Nova Scotia (N.S.), New Brunswick (N.B.), Quebec (QC), Ontario (ON), Manitoba (MN), Saskatchewan (SK), Alberta (AL), British Columbia (B.C.), the Yukon (YK), Northwest Territories and Nunavut (N.W.T.). See Table 2.8 for details.

Figure 2.8: Default Rates of Sample Groups by Rating



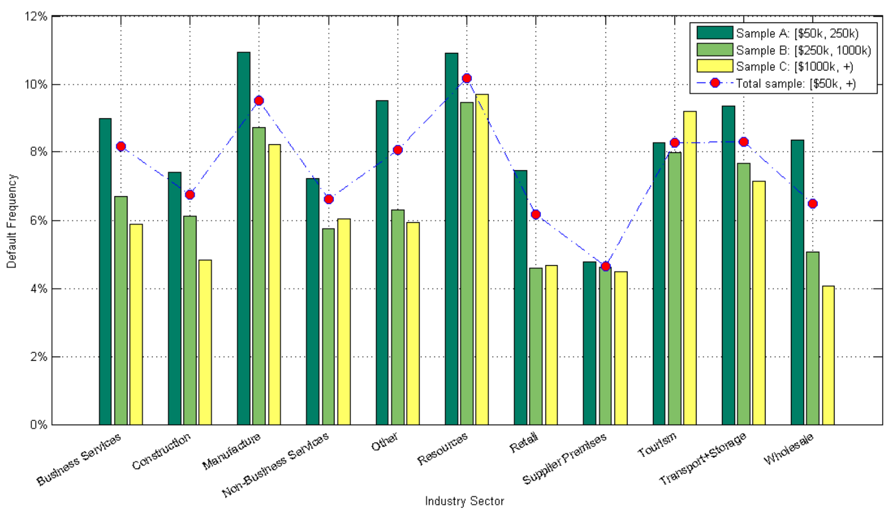
The figure shows average default rates by risk rating over 12 years (1998-2009), weighted by obligor years. Risk ratings range from 1.5 (least risky) to 4.5 (riskiest) with intervals of 0.5. To calculate the average default rates the summed number of defaulted obligor year over each calendar year is divided by the summed number of obligor year at the beginning of each calendar year. Four sample groups are classified by loan size: subsample A with loan size [$50k, $250k), subsample B with loan size [$250k, $1000k), subsample C with loan size [$1000k, $+\infty$) and the total sample.

Figure 2.9: Default Rates of Sample Groups Over Time



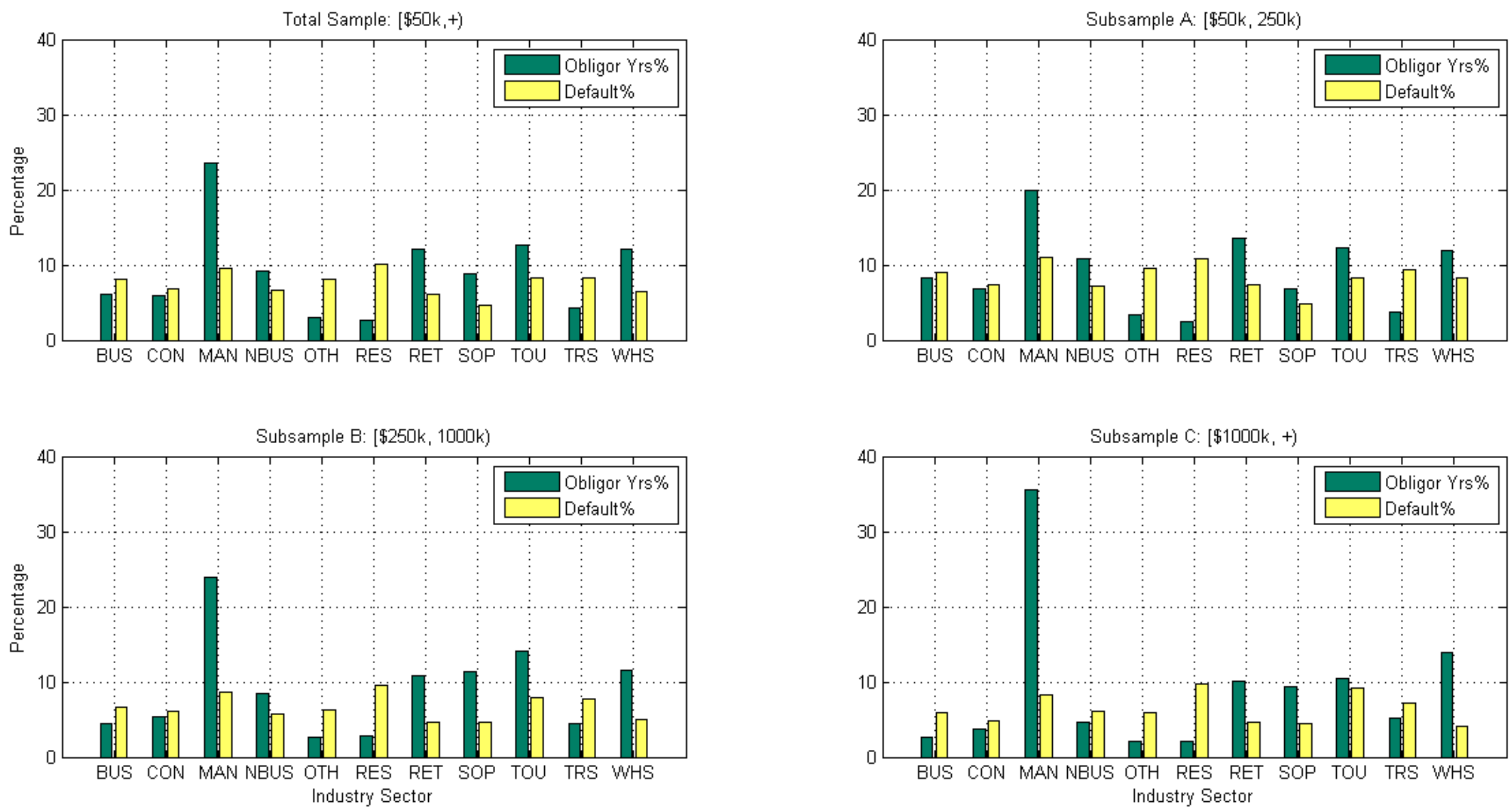
The figure exhibits the time series of annual default rates for the period from 1998 to 2009. Risk ratings range from 1.5 (least risky) to 4.5 (riskiest) with intervals of 0.5. To calculate the annual default rate, the number of defaulted obligor years over a given calendar year are divided by the number of obligor years at the beginning of the given calendar year. Four sample groups are classified by loan size: subsample A with loan size [$50k, $250k), subsample B with loan size [$250k, $1000k), subsample C with loan size [$1000k, $+\infty$) and the total sample.

Figure 2.10: Default Rates of Sample Groups by Sector



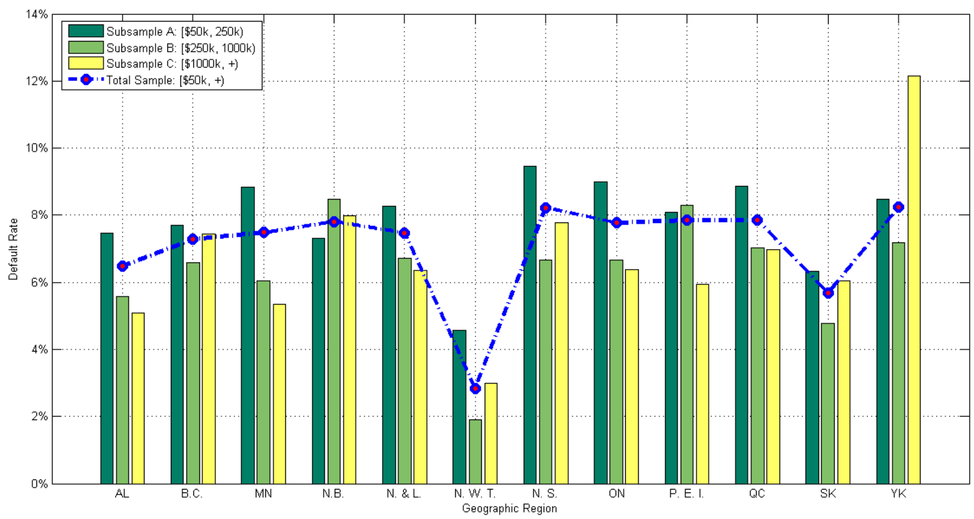
The figure shows the average default rates by industrial sectors over 12 years (1998-2009). To calculate the annual default rate, the number of defaulted obligor years over a given calendar year is divided by the number of obligor years at the beginning of the given calendar year. Four sample groups are classified by loan size: subsample A with loan size [$50k, $250k), subsample B with loan size [$250k, $1000k), subsample C with loan size [$1000k, $+\infty$) and the total sample.

Figure 2.11: Default Rates vs. Sector Composition of Sample Groups



The series of figures examines default rates versus proportion of obligors by dimensions of sector and sample groups over 12 years (1998 to 2009). To calculate the annual default rate, the number of defaulted obligor years over a given calendar year is divided by the number of obligor years at the beginning of the given calendar year. The industrial sectors, in alphabetical order, are as follows: Business Services (BUS); Construction (CON); Manufacturing (MAN); Non-Business Services (NBUS); Other (OTH); Resources (RES); Retail (RET); Supplier or Premises (SOP); Tourism (TOU); Transportation and Storage (TRS); Wholesale (WHS).

Figure 2.12: Default Rates of Sample Groups by Geographic Region



The figure shows default rates by geographical regions in different sample groups over 12 years (1998-2009). Four sample groups are classified by loan size: subsample A with loan size [$50k, $250k), subsample B with loan size [$250k, $1000k), subsample C with loan size [$1000k, $+\infty$) and the total sample. The regions, in alphabetical order, are as follows: Newfoundland & Labrador (N. & L.), Prince Edward Island (P.E.I.), Nova Scotia (N.S.), New Brunswick (N.B.), Quebec (QC), Ontario (ON), Manitoba (MN), Saskatchewan (SK), Alberta (AL), British Columbia (B.C.), the Yukon (YK), Northwest Territories and Nunavut (N.W.T.).

Table 2.1: PDs and Ratings Mapping (S&P and *Financing Company*)

S&P RR	Mean (%)	Std Dev (%)	FC RR	Mean(%)	Std Dev
AAA	0.00	0.00			
AA+	0.00	0.00			
AA	0.04	0.12			
AA-	0.06	0.15			
A+	0.10	0.19			
A	0.13	0.15			
A-	0.12	0.23			
BBB+	0.16	0.32			
BBB	0.28	0.26			
BBB-	0.50	0.46			
BB+	0.48	0.50			
BB	0.82	0.53			
BB-	1.45	1.76			
B+	2.86	2.18	**1**	2.78	1.05
			1.5	2.82	0.72
			2	4.54	0.99
			2.5	5.54	0.87
B	6.67	5.10	**3**	6.20	1.28
			3.5	7.85	1.30
			4	8.74	1.82
B-	10.21	7.48	**4.5**	10.27	1.27
			5	17.04	2.53
CCC/C	30.34	14.02	**5.5**	28.33	16.03

The table exhibits descriptive statistics for the *Financing Company* SME loans and Standard & Poors (S&P) rated corporate debt. Statistics are given by rating for the Mean, Standard Deviation (Std Dev). S&P statistics were measured over the same period of 1998 2009 as that for the *Financing Company* (FC). Source: Default, Transition, and Recovery: 2013 Annual Global Corporate Default Study and Rating Transitions. Standard & Poor's RatingsDirect on the Global Credit Portal: 2013.

Table 2.2: Rating Distributions of Portfolio by Loan Size

Rating	Loan size (K$)									
	($0,$50k)		[$50k,$250k)		[$250k,$1000k)		[1000k$, $+\infty$)		Total	
	No.	%	No.	%	No.	%	No.	%	No.	%
1	442	1.78	2,645	2.65	2,794	3.78	2,122	7.47	8,003	3.53
1.5	805	3.24	5,854	5.87	6,925	9.36	3,589	12.63	17,173	7.57
2	1,311	5.28	8,639	8.67	7,217	9.76	3,708	13.05	20,875	9.20
2.5	1,352	5.45	8,548	8.58	8,182	11.06	3,945	13.89	22,027	9.71
3	1,965	7.92	12,390	12.43	12,027	16.26	4,987	17.55	31,369	13.83
3.5	1,958	7.89	9,193	9.22	9,137	12.36	3,443	12.12	23,731	10.46
4	5,390	21.73	18,874	18.94	14,149	19.13	3,876	13.64	42,289	18.64
4.5	6,221	25.08	22,508	22.58	11,909	16.10	2,353	8.28	42,991	18.95
5	5,341	21.53	10,872	10.91	1,474	1.99	296	1.04	17,983	7.93
5.5	23	0.09	137	0.14	137	0.19	89	0.31	386	0.17
Total	24,808	10.94	99,660	43.94	73,951	32.60	28,408	12.52	226,827	100.00
Exposure	515	0.66	8720	11.13	25300	32.33	43800	55.88	78000	100.00

The table describes the rating distributions of the *Financing Company* portfolio by loan size. The data is based on obligor-year observations over 12 years (1998-2009). The portfolio consists of obligors rated from 1.5 (least risk) to 4.5 (riskiest) with 0.5 rating intervals. Exposure for each loan size bucket is cumulative loan amount of obligor-year observations over 12 years and the measure unit is in million $.

Table 2.3: Rating Distributions of Sample Groups

Rating	Loan size (K$)							
	Subsample A		Subsample B		Subsample C		Total Sample	
	No.	%	No.	%	No.	%	No.	%
1.5	8,499	8.53	9,719	13.14	5,711	20.10	23,929	11.84
2	8,639	8.67	7,217	9.76	3,708	13.05	19,564	9.68
2.5	8,548	8.58	8,182	11.06	3,945	13.89	20,675	10.23
3	12,390	12.43	12,027	16.26	4,987	17.55	29,404	14.56
3.5	9,193	9.22	9,137	12.36	3,443	12.12	21,773	10.78
4	18,874	18.94	14,149	19.13	3,876	13.64	36,899	18.27
4.5	33,517	33.63	13,520	18.28	2,738	9.64	49,775	24.64
Total	99,660	49.33	73,951	36.61	28,408	14.06	202,019	100.00
Exposure	8720	11.21	25300	32.51	43800	56.28	77820	100.00

The table describes the rating distributions of the *Financing Company* portfolio in four sample groups, including subsample *A* with loan size [\$50k, \$250k), subsample *B* with loan size [\$250k, \$1000k) ,subsample *C* with loan size [\$1000k, $+\infty$) and total sample. The data is based on obligor-year observations over 12 years (1998-2009). The portfolio consists of obligors rated from 1.5 (least risk) to 4.5 (riskiest) with 0.5 rating intervals.

Table 2.4: Rating Distributions of Sample Groups Over Time (%)

Rating	Calendar Year											
	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Total Sample												
1.5	7.79	5.39	7.47	10.78	10.94	10.88	11.09	10.17	10.36	13.37	15.43	16.32
2	7.62	6.25	6.91	7.72	8.08	8.70	8.86	9.03	9.21	10.83	11.64	12.66
2.5	9.25	7.26	7.87	8.68	9.45	9.99	10.41	10.96	11.09	11.01	10.81	11.28
3	19.03	18.24	16.49	15.96	15.81	15.39	14.57	14.81	14.04	12.43	12.34	12.70
3.5	18.04	14.47	11.54	10.30	9.51	10.48	10.21	10.13	10.14	10.02	10.26	9.85
4	26.21	27.75	24.08	22.11	21.27	19.63	18.47	18.01	17.81	14.63	13.63	12.97
4.5	12.07	20.63	25.64	24.47	24.95	24.94	26.39	26.89	27.35	27.71	25.88	24.23
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Subsample *A*												
1.5	6.11	4.41	5.54	8.46	8.01	7.88	8.12	7.03	7.22	9.24	10.51	11.41
2	6.69	5.28	5.66	5.96	6.86	7.15	7.18	7.79	8.09	9.25	10.53	12.34
2.5	7.96	6.12	5.84	6.58	7.72	7.75	8.16	8.75	8.73	9.00	9.59	10.46
3	17.89	15.73	13.93	13.71	13.37	12.58	12.00	12.19	11.50	10.28	10.58	11.28
3.5	17.61	14.00	10.77	8.96	8.26	9.32	8.95	8.46	8.36	8.19	8.00	7.50
4	28.80	29.82	27.24	24.97	22.62	20.85	18.80	18.60	17.74	14.53	13.51	12.54
4.5	14.94	24.64	31.02	31.36	33.16	34.47	36.79	37.19	38.37	39.53	37.27	34.47
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Subsample *B*												
1.5	8.66	5.83	7.73	11.56	11.82	12.25	12.23	11.96	12.25	15.25	18.33	18.82
2	7.94	6.68	7.59	8.55	8.22	9.30	9.71	9.37	9.88	11.28	11.74	11.85
2.5	10.14	7.78	9.44	9.28	10.52	11.30	11.22	11.91	12.19	12.23	11.40	11.82
3	20.14	20.02	17.70	17.22	17.20	16.75	16.18	16.35	15.51	14.43	14.49	14.77
3.5	18.62	14.97	11.66	11.69	10.66	11.25	11.19	11.46	11.89	12.45	12.84	12.42
4	24.85	26.42	22.51	20.99	21.47	19.69	19.38	19.01	18.96	16.34	15.34	14.62
4.5	9.65	18.30	23.37	20.72	20.11	19.47	20.08	19.94	19.33	18.02	15.86	15.70
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Subsample *C*												
1.5	13.74	8.21	14.12	15.99	17.49	15.89	16.66	14.85	15.21	22.08	23.86	24.79
2	11.41	9.01	9.43	10.95	11.22	11.38	11.37	11.47	10.66	14.48	14.06	14.15
2.5	12.83	10.46	10.62	13.80	11.54	12.90	14.62	14.74	15.24	14.38	12.65	11.82
3	20.51	22.93	22.08	19.12	18.91	19.58	17.49	18.19	17.78	14.45	12.62	11.97
3.5	17.58	14.16	13.63	10.18	9.75	11.38	10.95	11.43	11.12	10.18	11.40	11.12
4	15.86	21.08	14.81	13.68	14.70	14.33	13.99	12.69	14.43	10.46	9.52	10.13
4.5	4.75	8.61	9.43	9.35	9.64	8.06	8.87	10.41	9.99	9.12	8.52	8.34
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

The table 2.4 describes the rating distributions history of the *Financing Company* portfolio in terms of obligor-year observations over 12 years (1998-2009). The portfolio consists of obligors rated from 1.5 (least risk) to 4.5 (riskiest) with 0.5 rating intervals. Table is segregated into four panels for total sample, subsample A with loan size [$50k, $250k), subsample B with loan size [$250k, $1000k), and subsample C with loan size [$1000k, $+\infty$).

Table 2.5: Sector Composition of Sample Groups

	Loan size (K$)							
Rating	Subsample A		Subsample B		Subsample C		Total Sample	
	No.	%	No.	%	No.	%	No.	%
BUS	9,904	9.15	4,221	5.35	1,005	3.34	15,130	6.97
CON	8,761	8.10	5,744	7.29	1,591	5.29	16,096	7.41
MAN	23,991	22.17	21,647	27.46	11,801	39.24	57,439	26.46
NBS	12,040	11.13	6,950	8.82	1,428	4.75	20,418	9.40
OTH	3,062	2.83	2,019	2.56	638	2.12	5,719	2.63
RES	2,914	2.69	2,535	3.22	710	2.36	6,159	2.84
RET	14,646	13.53	8,860	11.24	3,319	11.04	26,825	12.36
SOP	5,090	4.70	5,906	7.49	1,867	6.21	12,863	5.92
TOU	12,241	11.31	10,043	12.74	2,926	9.73	25,210	11.61
TRS	3,768	3.48	3,299	4.19	1,517	5.04	8,584	3.95
WHS	11,793	10.90	7,603	9.65	3,275	10.89	22,671	10.44
Total	108,210	100.00	78,827	100.00	30,077	100.00	217,114	100.00

The table describes the sector composition of the *Financing Company* portfolio in four sample groups, including subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k) ,subsample C with loan size [\$1000k, $+\infty$) and total sample. The data is based on obligor-year observations over 12 years (1998-2009). The portfolio consists of obligors rated from 1.5 (least risk) to 4.5 (riskiest) with 0.5 rating intervals. The industrial sectors, in alphabetical order, are as follows: Business Services (BUS); Construction (CON); Manufacturing (MAN); Non-Business Services (NBUS); Other (OTH); Resources (RES); Retail (RET); Supplier or Premises (SOP); Tourism (TOU); Transportation and Storage (TRS); Wholesale (WHS).

Table 2.6: Rating Distributions of Sample Groups by Sectors (%)

Rating	BUS	CON	MAN	NBS	OTH	RES	RET	SOP	TOU	TRS	WHS
Total Sample											
1.5	13.36	13.18	15.25	10.24	10.07	12.20	11.48	4.58	6.64	8.82	14.67
2	10.82	10.54	12.00	8.33	8.94	9.74	9.32	3.79	6.25	9.24	11.71
2.5	10.80	10.90	11.44	10.30	10.12	9.63	10.21	5.63	7.67	10.25	12.00
3	14.95	15.15	15.67	13.22	14.17	14.89	14.53	11.43	13.39	15.84	14.95
3.5	9.50	10.91	11.18	11.15	9.80	10.77	10.35	11.12	10.70	12.22	10.30
4	14.54	15.61	15.76	17.93	18.07	17.10	18.73	30.49	23.87	18.82	15.57
4.5	26.03	23.70	18.69	28.84	28.83	25.66	25.38	32.95	31.47	24.81	20.80
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Subsample *A*											
1.5	10.65	10.26	10.54	7.16	7.67	8.30	8.28	3.45	5.21	5.87	9.97
2	10.76	9.46	10.26	8.00	8.42	8.53	8.47	2.75	5.59	7.56	10.35
2.5	10.17	9.83	9.31	8.58	8.53	8.27	8.06	3.61	6.38	7.71	10.32
3	14.02	12.35	13.33	11.79	11.56	11.66	12.54	8.49	11.39	13.51	12.80
3.5	7.93	9.56	9.97	9.82	8.60	10.25	8.45	9.68	8.90	9.81	8.80
4	14.35	16.33	17.13	18.13	17.09	17.75	20.47	32.23	23.27	18.62	17.73
4.5	32.14	32.21	29.46	36.53	38.14	35.24	33.72	39.79	39.25	36.92	30.02
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Subsample *B*											
1.5	17.76	14.82	15.99	13.62	10.90	13.71	13.45	4.17	6.52	8.68	18.40
2	10.25	11.36	12.53	8.10	9.43	10.18	9.51	3.55	6.01	8.78	12.43
2.5	11.60	11.97	12.37	12.07	11.84	10.13	11.73	5.69	8.20	11.03	12.73
3	16.19	18.54	17.21	14.97	16.98	17.03	16.14	12.04	14.74	18.21	17.27
3.5	12.58	12.70	12.41	13.19	10.32	11.86	12.55	11.49	12.13	14.87	11.42
4	15.53	15.36	16.46	18.45	20.55	18.20	18.40	31.20	25.46	20.01	14.91
4.5	16.09	15.25	13.03	19.60	19.97	18.89	18.22	31.86	26.94	18.43	12.83
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Subsample *C*											
1.5	20.99	23.00	23.18	19.20	18.57	22.32	19.84	8.94	13.09	16.23	22.26
2	13.85	13.36	14.48	12.19	9.78	12.97	12.43	7.39	9.85	14.27	14.74
2.5	13.54	12.83	13.92	15.84	12.11	13.27	15.34	10.83	11.27	14.69	16.13
3	18.68	18.05	17.49	16.50	17.41	20.21	18.71	17.44	17.07	16.37	16.98
3.5	11.65	11.79	11.32	12.19	13.76	9.05	12.65	13.83	13.31	12.37	12.87
4	12.17	12.64	11.80	13.80	14.76	10.71	12.18	23.56	20.86	16.73	9.65
4.5	9.13	8.34	7.81	10.29	13.60	11.46	8.84	18.00	14.54	9.35	7.37
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

The table describes the rating distributions of the *Financing Company* portfolio across sectors in terms of obligor-year observations over 12 years (1998-2009). The portfolio consists of obligors rated from 1.5 (least risk) to 4.5 (riskiest) with 0.5 rating intervals. The table is segregated into four Panels for total sample, subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), and subsample C with loan size [\$1000k, $+\infty$). The industrial sectors, in alphabetical order, are as follows: Business Services (BUS); Construction (CON); Manufacturing (MAN); Non-Business Services (NBUS); Other (OTH); Resources (RES); Retail (RET); Supplier or Premises (SOP); Tourism (TOU); Transportation and Storage (TRS); Wholesale (WHS).

Table 2.7: Geographical Composition of Sample Groups

Rating	Loan size (K$)							
	Subsample A		Subsample B		Subsample C		Total Sample	
	No.	%	No.	%	No.	%	No.	%
Alberta	7,892	7.29	5,116	6.49	2,329	7.74	15,337	7.06
British Columbia	12,983	12.00	7,922	10.05	2,599	8.64	23,504	10.83
Manitoba	3,069	2.84	1,903	2.41	535	1.78	5,507	2.54
New Brunswick	4,672	4.32	3,089	3.92	1,078	3.58	8,839	4.07
N.& L.	5,428	5.02	4,130	5.24	866	2.88	10,424	4.80
N.W. Territories	239	0.22	466	0.59	177	0.59	882	0.41
Nova Scotia	3,496	3.23	2,393	3.04	654	2.17	6,543	3.01
Ontario	32,638	30.16	23,281	29.53	9,420	31.32	65,339	30.09
P. E. I.	579	0.54	572	0.73	199	0.66	1,350	0.62
Quebec	34,086	31.50	27,484	34.87	11,600	38.57	73,170	33.70
Saskatchewan	2,745	2.54	2,082	2.64	525	1.75	5,352	2.47
Yukon	383	0.35	389	0.49	95	0.32	867	0.40
Total	108,210	100.00	78,827	100.00	30,077	100.00	217,114	100.00

The table describes the geographical composition of *Financing Company* portfolio in four sample groups, including subsample *A* with loan size [$50k, $250k), subsample *B* with loan size [$250k, $1000k) ,subsample *C* with loan size [$1000k, $+\infty$) and the total sample. The data is based on obligor-year observations over 12 years (1998-2009). The portfolio consists of obligors rated from 1.5 (least risk) to 4.5 (riskiest) with 0.5 rating intervals. The regions include Newfoundland & Labrador (N. & L.), Prince Edward Island (P.E.I.), Nova Scotia (N.S.), New Brunswick (N.B.), Quebec (QC), Ontario (ON), Manitoba (MN), Saskatchewan (SK), Alberta (AL), British Columbia (B.C.), the Yukon (YK), Northwest Territories and Nunavut (N.W.T.).

Table 2.8: Rating Distributions of Sample Groups by Geography (%)

Rating	AL	B.C.	MN	N.B.	N. & L.	N. W. T.	N. S.	ON	P. E. I.	QC	SK	YK
Total Sample												
	Geographic Region											
1.5	12.51	10.07	10.10	10.70	11.50	12.91	8.07	10.75	4.31	14.20	10.08	9.51
2	9.94	8.78	9.57	7.46	8.74	8.02	8.15	9.36	5.58	10.85	10.47	5.51
2.5	10.76	8.81	10.14	8.19	9.89	11.86	8.67	9.97	6.46	11.41	9.41	6.88
3	13.31	13.46	13.99	13.02	14.03	17.91	12.88	14.00	11.00	16.22	13.47	14.77
3.5	9.68	9.97	10.59	11.31	10.35	8.49	11.53	10.39	11.24	11.62	9.88	13.14
4	17.47	20.02	17.38	21.15	19.17	21.63	19.24	18.02	24.16	17.26	19.74	24.53
4.5	26.33	28.89	28.24	28.18	26.32	19.19	31.46	27.51	37.24	18.44	26.96	25.66
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Subsample *A*												
1.5	8.20	7.42	7.68	8.55	8.42	7.42	6.59	6.89	3.91	11.16	6.46	7.18
2	8.45	8.06	8.28	6.61	8.50	3.06	7.43	8.01	4.84	10.16	9.21	4.89
2.5	8.99	7.77	7.86	6.84	8.62	6.55	7.59	7.80	5.77	10.08	7.89	6.32
3	11.19	12.23	12.03	9.66	12.95	12.66	10.60	11.70	7.64	14.12	12.03	14.08
3.5	8.30	8.33	8.24	9.08	8.82	8.73	10.07	8.74	8.94	10.29	9.28	13.22
4	18.05	19.93	16.77	23.09	19.63	28.38	18.44	19.02	22.53	17.99	19.88	23.56
4.5	36.83	36.26	39.14	36.16	33.05	33.19	39.29	37.83	46.37	26.20	35.24	30.75
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Subsample *B*												
1.5	15.45	11.30	11.42	11.34	13.85	13.94	8.56	12.22	2.27	14.94	12.00	12.60
2	10.67	8.48	9.59	6.37	8.88	9.37	8.61	9.70	4.36	10.71	10.89	6.03
2.5	11.62	9.11	11.09	8.86	10.79	13.51	9.32	11.23	6.44	11.95	11.04	5.75
3	15.45	14.21	16.69	16.46	14.52	18.74	15.79	15.83	11.74	17.84	14.72	13.15
3.5	10.71	11.88	14.80	14.39	11.82	8.50	13.07	11.86	13.26	13.05	10.33	12.05
4	18.15	22.18	19.35	20.76	19.34	18.74	20.83	18.77	27.08	18.04	20.21	25.75
4.5	17.94	22.84	17.07	21.81	20.80	17.21	23.82	20.39	34.85	13.46	20.82	24.66
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
Subsample *C*												
1.5	20.34	19.38	18.86	18.07	19.08	17.44	13.98	20.10	11.11	21.16	21.12	5.81
2	13.29	13.22	16.70	14.12	9.54	11.05	10.24	13.06	11.11	13.14	15.34	5.81
2.5	14.73	12.97	19.45	12.04	13.29	14.53	11.87	14.18	8.47	13.98	10.76	13.95
3	15.63	17.26	15.32	17.77	18.24	22.67	14.15	17.23	18.52	18.43	15.94	24.42
3.5	11.99	12.20	8.64	12.14	12.68	8.14	13.50	12.33	12.17	12.06	11.16	17.44
4	14.05	13.91	13.75	13.92	15.58	20.35	17.56	12.81	20.63	13.32	17.13	23.26
4.5	9.97	11.06	7.27	11.94	11.59	5.81	18.70	10.30	17.99	7.92	8.57	9.30
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

The table describes the rating distributions of *Financing Company* portfolio across geographical regions in terms of obligor-year observations over 12 years (1998-2009). The portfolio consists of obligors rated from 1.5 (least risk) to 4.5 (riskiest) with 0.5 rating intervals. The table is segregated into four panels for total sample, subsample A with loan size [$50k, $250k), subsample B with loan size [$250k, $1000k), and subsample C with loan size [$1000k, $+\infty$). The regions, in alphabetical order, are as follows: Newfoundland & Labrador (N. & L.), Prince Edward Island (P.E.I.), Nova Scotia (N.S.), New Brunswick (N.B.), Quebec (QC), Ontario (ON), Manitoba (MN), Saskatchewan (SK), Alberta (AL), British Columbia (B.C.), the Yukon (YK), Northwest Territories and Nunavut (N.W.T.).

Table 2.9: Default Rates of Sample Groups by Rating (%)

Risk Rating	Loan size			
	Total Sample	Subsample A	Subsample B	Subsample C
1.5	2.96	3.60	2.41	3.00
2	4.77	5.21	4.36	4.61
2.5	5.65	6.63	4.83	5.31
3	6.22	6.59	5.80	6.38
3.5	7.91	7.61	7.89	8.79
4	8.30	7.89	8.48	9.66
4.5	11.60	12.22	10.04	12.05
Average	7.57	8.50	6.70	6.67

The table shows average default rates by risk rating over 12 years (1998-2009), weighed by obligor years. Risk ratings range from 1.5 (least risky) to 4.5 (riskiest) with intervals of 0.5. To calculate the average default rates the summed number of defaulted obligor year over each calendar year is divided by the summed number of obligor year at the beginning of each calendar year. Four sample groups are classified by loan size: subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and the total sample.

Table 2.10: **Default Rates of Sample Groups over Time (%)**

Risk Rating	Loan size			
	Total Sample	Subsample A	Subsample B	Subsample C
1998	6.57	6.79	6.77	4.61
1999	8.27	8.88	8.17	5.76
2000	7.70	7.73	7.98	6.65
2001	7.15	7.20	6.99	7.53
2002	7.48	7.76	7.21	7.36
2003	6.42	6.87	5.73	6.94
2004	7.43	8.34	6.69	6.54
2005	7.36	8.53	6.18	6.64
2006	6.43	7.64	4.95	6.06
2007	7.04	8.51	5.81	4.88
2008	9.42	11.09	7.66	7.69
2009	8.89	10.33	7.09	7.98
Average	7.57	8.50	6.70	6.67

The table shows the time series of annual default rates for the period from 1998 to 2009. Risk ratings range from 1.5 (least risky) to 4.5 (riskiest) with intervals of 0.5. To calculate the annual default rate, the number of defaulted obligor years over a given calendar year is divided by the number of obligor years at the beginning of the given calendar year. Four sample groups are classified by loan size: subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and the total sample.

Table 2.11: Default Rates of Sample Groups by Sector (%)

Sector	Loan size			
	Total Sample	Subsample A	Subsample B	Subsample C
BUS	8.16	8.98	6.71	5.89
CON	6.74	7.42	6.11	4.82
MAN	9.52	10.92	8.71	8.22
NBS	6.63	7.22	5.75	6.04
OTH	8.07	9.52	6.30	5.94
RES	10.17	10.90	9.46	9.69
RET	6.17	7.45	4.59	4.67
SOP	4.66	4.78	4.62	4.47
TOU	8.26	8.27	8.00	9.21
TRS	8.31	9.37	7.66	7.15
WHS	6.49	8.36	5.06	4.06
Average	7.57	8.50	6.70	6.67

The table shows the average default rates by industrial sectors for the period from 1998 to 2009. To calculate the annual default rate, the number of defaulted obligor years over a given calendar year is divided by the number of obligor years at the beginning of the given calendar year. Four sample groups are classified by loan size: subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and the total sample. The industrial sectors, in alphabetical order, are as follows: Business Services (BUS); Construction (CON); Manufacturing (MAN); Non-Business Services (NBUS); Other (OTH); Resources (RES); Retail (RET); Supplier or Premises (SOP); Tourism (TOU); Transportation and Storage (TRS); Wholesale (WHS).

Table 2.12: Default Rates of Sample Groups by Geography (%)

Risk Rating	Loan size			
	Total Sample	Subsample A	Subsample B	Subsample C
Alberta	6.47	7.46	5.58	5.09
British Columbia	7.28	7.69	6.59	7.43
Manitoba	7.48	8.83	6.04	5.35
New Brunswick	7.81	7.32	8.47	7.99
N. & L.	7.47	8.27	6.72	6.35
N. W. Territories	2.84	4.57	1.89	2.99
Nova Scotia	8.23	9.45	6.66	7.77
Ontario	7.77	8.99	6.66	6.38
P. E. I.	7.85	8.08	8.30	5.95
Quebec	7.85	8.86	7.02	6.99
Saskatchewan	5.68	6.32	4.76	6.05
Yukon	8.23	8.46	7.18	12.16
Average	7.57	8.50	6.70	6.67

The table shows the average default rates by geographical regions over 12 years (1998-2009). To calculate the annual default rate, the number of defaulted obligor years over a given calendar year is divided by the number of obligor years at the beginning of the given calendar year. Four sample groups are classified by loan size: subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and the total sample. The regions, in alphabetical order, are as follows: Newfoundland & Labrador (N. & L.), Prince Edward Island (P.E.I.), Nova Scotia (N.S.), New Brunswick (N.B.), Quebec (QC), Ontario (ON), Manitoba (MN), Saskatchewan (SK), Alberta (AL), British Columbia (B.C.), the Yukon (YK), Northwest Territories and Nunavut (N.W.T.).

Chapter 3

Estimation of Credit Migration Matrices

3.1 Introduction

Encouraged by Basel II, many financial institutions have based their credit risk management on an internal rating system. As credit rating migration matrices, which characterize the expected changes on the credit quality of obligors, have become major determinants for portfolio risk assessment, credit derivative pricing and other risk management applications, the accurate and proper estimation of rating transitions are important. However, several features of internal rating data — incomplete observations (due to non-continuous monitoring), varying time intervals between rating assessments and the high percentage of non-rated (NR) obligors — motivate us to compare and analyze different methods to accommodate such data.

This chapter is an empirical study of several multi-state Markov chain methods for estimating credit migration matrices. These methods are then applied to the study of the internal rating system of the *Financing Company* SME loan portfolio presented in the last chapter. Specifically, we explore three approaches: the industry standard "Cohort" method, the academically well known "Duration" method and the survival analysis based "MT-Duration" (Mixed-Time observation Duration) method. We now provide an overview of the three approaches; formal details appears in subsequent sections.

The first empirical credit rating migration model was developed by Jarrow et al. (1997) who used firm data to construct a matrix of credit rating transitions probabilities. The matrix is based on a discrete-time homogeneous Markov chain, where each element represents the frequency of a particular rating migration. Simply put, each entry of the matrix represents the probability of a specific rating transition, and is calculated by dividing the number of companies that moved from one state to another by the total number of companies in the initial rating category. The external rating agencies (Moody's, S&P, Fitch and etc.) and financial institutions use the same method, known now as the cohort method. Although convenient, the cohort method is found to be inefficient due to its discrete-time setting. First of all, the accurate estimation of default probabilities for the higher rating grades, such as large corporations with a rating of Aaa or Aa, is difficult due to the scarcity of data over short periods. The problem is even more apparent when individual banks attempt to utilize their own internal rating data. In these cases, a straightforward estimation based on the simple average often results in deriving zero default probabilities. Second, the methodology of the cohort method ignores indirect defaults (i.e., default through a sequence of downgrades) for all rating levels. Third, we also notice that the frequency of rating review is not always the same for every obligor in individual banks. Machauer and Weber (1998) report that the frequency of reviews may be based on the obligor's current rating or its collateral. The time span between two consecutive monitorings is not always one year. Therefore the different frequencies of rating reviews may bring about efficiency loss for the Cohort estimator.

Recent studies based on the modeling of continuous-time rating transition matrices (i.e., the Duration method) seem to provide a potential solution to these problems. The key idea here is to capture the possibility of successive downgrades of an obligor from higher rating grades toward the lower rating grades where defaults occur more frequently. For example, if transitions from Aaa to A and transitions from A to default are observed within a period, then one can consider the possibility that an obligor with Aaa might default after successive downgrades, even without observing direct transitions from Aaa to default. The continuous time approach incorporates the possibility of successive downgrades leading to default in such a way that a very slight probability of default can be captured. Lando and Skodeberg (2002) provide two continuous-time estimators for credit rating matrices, which differ in terms of their assumption regarding transition intensities. Specifically, one estimator, called the Aalen-Johansen estimator, allows transition intensities to be time inhomogeneous (i.e. time-varying) over the business cycle. While the other estimators assume that transition intensities are time homogeneous (i.e. time-invariant) and thus estimates a so-called infinitesimal generator matrix in continuous time Markov chains. Their empirical studies show that both of these estimators generate non-zero values for default and migrating probabilities, which the standard multinomial method (i.e.,the Cohort method) does not capture. Later, developing a new distance metric for transition matrices, Jafry and Schuermann (2004) compare all theses methods (Cohort, Duration-homogeneous, Duration-inhomogeneous) statistically via a bootstrap approach. They also investigated the economic consequences of using the three approaches via contrasting economic capital calculations. Overall, the study shows that the default probabilities estimated via Duration are higher than the estimates via Cohort for the extreme ratings (highest and lowest) but much lower for the middle ratings. In the Duration approach, the differences between homogeneous and non-homogeneous assumptions are less pronounced.

One concern with the continuous-time approach is that most of the previous literature is based on continuously observed transition data from external rating agencies. There is at least one fundamental difference between external and internal rating data which questions the applicability of the Duration method in internal rating systems: internal ratings are not continuously monitored. In applications, a high-frequency database of intra-year transition records is costly at the individual bank level. Indeed, the data frequency of internal rating systems for individual banks is either annual or bi-annual in most cases. Accordingly, rating changes described in a continuous time settings could pass unnoticed if monitoring is infrequent.

Mählmann (2006) proposed a solution to the issue by a Maximum Likelihood (ML) procedure derived from earlier work of Kay (1986) in the literature of survival analysis. In the ML procedure, the key idea is to account for the interval-censoring time issue and the censored states issue to treat the likelihood function in a continuous-time setting. Specifically, the exact transition time is rare to witness, but could be known to occur within a particular interval. We refer to this time issue as interval-censoring. The ML procedure uses a different maximum likelihood function from that of the Duration method to deal with this issue. Moreover, states may be censored as well as event times. For example, if an obligor does not default within the period studied and is not present at the end of study, it is in a censored state at the end of period studied (i.e., it is alive with unknown rating); If an obligor defaults, the rating grade on the previous instant before default is unknown and we set it as censored state. The contribution of censored state to the likelihood is to summarize the transition probabilities over all possible states that could have been visited. Mählmann (2006) compared the industry standard Cohort method and ML procedure with data from a German bank, 85% of whose borrowers are SMEs (The definition of SMEs is annual turnover equal or less than Euro 50 Millions). The differences between the transition matrices estimated by the two methods are shown to be significant statistically and economically.

The Cohort method overestimates default risk compared with the ML approach. The paper doesn't clarify completely the theoretical difference between Cohort, Duration and proposed ML approaches. Also, it doesn't provide empirical differences between Duration and the ML procedure. Meanwhile, due to the high percentage of one-year fixed observation time (92.77%), the paper did not fully exhibit the advantages of the ML procedure over the Cohort method.

Indeed, it has been noticed that this ML procedure is actually the homogeneous multi-state Markov model (HMM) for mixed discrete-continuous time observations data in the literature of survival analysis. Commenges (2002) provides an explicit description about the HMM model under different observation patterns. As this paper points out, the observations in clinical trials are always incomplete because we cannot make observations in continuous time but only at a finite number of distinct times. This leads to interval-censored observations. The most common pattern of observations is in fact a mixing of discrete and continuous time observations. This is because most often the clinical status is observed at discrete time and the death can be observed at (nearly) exact time of occurrence (or at continuous time). Drawing parallels between the process of credit rating review and clinical trials involving treatment and response, we see that both cases deal with time-to-event data. The state of identities moves from one (e.g. high rating/"healthy") to another (e.g. low rating/"sick") with an absorbing state (i.e., death/default). Therefore, the issues of an internal rating system (i.e., discrete observed rating data with irregular frequency) that ML procedure tries to solve are due to mixed discrete-continuous observation data. Since the Duration method is HMM for the continuous time observations, we refer to this ML procedure as the Mixed-Time Duration (a MT-Duration) method in the thesis to differentiate the methods. Since Duration and MT-Duration are both intensity-based estimation procedure under continuous-time modeling, it is natural to question: what is the extent of the difference between Duration and MT-Duration.

Based on these recent advances, this chapter discusses and compares the three competing estimation methods of transition matrices (i.e., "Cohort", "Duration", and "MT-Duration") theoretically and empirically in the rating settings of the *Financing Company*. It is the first attempt to investigate the differences between "Duration" and "MT-Duration" methods in the literature of credit risk. The theoretical part of this chapter studies the drivers of differences among the three competing models under the assumption that credit quality dynamics follow a time-homogeneous first-order Markov process. In the empirical part of the chapter, we ask three questions: (i) what are the relevant features of the *Financing*

Company's internal rating data and its review process? (ii) are differences among the three methods statistically significant? (iii) even if the differences are statistically significant, are they economically significant? (iv) do the assumptions behind the three models hold for the SME rating migration process?

Concerning the first question, we found that rating review time of the *Financing Company* might be more than one year considering the reviewing cost or less than one year if some credit event triggers the rating review. Non-continuous monitoring and the different review frequency process for obligors suggest that the application of Cohort and Duration methods could be inappropriate. We use the three competing methods to compute credit migration matrices from *Financing Company*'s SME loan portfolio covering the period 1998-2009. We then compare the resulting differences, both statistically through a mobility-based metric and risk-sensitive metrics, and economically with a credit portfolio model. We show that results are indeed statistically and economically different.

Lastly, we analyze the Markovian behavior of our SME data. We find evidence for SMEs' rating reversal activity which is in line with studies with internal rating data but contrary to the rating drift seen in the external rating data. Using an Aalen-Johansen estimator, we find only a marginal difference between the continuous-time migration matrices estimates with and without the assumption of time-homogeneity for a one-year time horizon, which is similar to the studies in Lando and Skodeberg (2002) with external rating data.

The outline of this chapter is as follows. Section 2 describes the dataset and related features. Section 3 provides a detailed description of the theoretical estimation methodologies described above. Section 4 contains the empirical results of estimates based on the three competing methods. The formal comparison between transition matrices is analyzed using both mobility-based and risk-sensitive metrics. Section 5 measures the economic relevance of differences via the credit VaRs of loss distributions. Section 6 examines the properties of Markov and time-homogeneity. The chapter concludes with section 7.

3.2 The Data: Rating Frequency

The data comes from the internal rating system of *Financing Company* whose rating system and sample data are described throughly in Chapter 2. Briefly, the SME loan portfolio covers 12 years of ratings history from 1998 to 2009. The final sample data covers obligors with loan size above \$ 50k under a seven-position rating system we have constructed as described in Chapter 2. The seven-position numerical rating system reflects credit quality deterioration from 1.5 to 4.5 in intervals of 0.5 with 1.5 representing the safest rating and 4.5 representing the riskiest one. Based on the impact of loan size and characteristics of the rating sytem in *Financing Company*, we will investigate the complete sample and 3 sub-samples with loan size thresholds of \$250k and \$1000k. In all, the sample data contains 202,019 obligor years with 15,095 defaulted and 45,087 non-defaulted obligors.

3.2.1 Irregular Pattern of Observation Time

Longitudinal data obtained from monitoring credit quality progression are often incomplete in some way. Usually obligors are monitored at intermittent (discrete) follow-up reviews, at which monitoring information is collected. A fixed observation schedule may be specified in advance; e.g., annual or semi-annual reviews. Difficulties arise when we are faced with a rather irregular pattern of times at which observations are taken that might also vary from obligor to obligor. Several academic studies (see Machauer and Weber (1998), Diamond (1991), Blackwell and Winters (1997) and Mählmann (2006)) also provide evidence that for individual banks two consecutive monitoring processes are not always equally spaced and that monitoring time depends on borrower-specific characteristics, such as the borrower's reputation, her relationship with the bank, loan quality, etc. Therefore, it is necessary to look closely at the observation pattern of our rating data.

Similar to other banks, the *Financing Company* studied in this thesis monitors obligors only on an annual base due to the high expenditure associated with more frequent monitoring. In fact, an annual rating review is strictly executed only for large loans with amounts over \$1 million (sub-sample C) but not for smaller loans. The time between rating reviews for small loans may be over one year. Moreover, credit information from an external credit bureau might trigger a rating review as well. The rating data is recorded in monthly units. For the months without monitoring, the records of the previous rating status are kept. Since the monitoring time is not available in the database, we could only investigate the intensity of rating changes to have a rough picture of the ratings pattern.

Table 3.1 summarizes the time interval distribution of rating changes. We take the year 2004 to separate the data into two periods to track the possible policy change of monitoring after 2004. Since the rating review is a process, the recording of rating change might not capture exactly one-year time intervals. Accordingly, the time interval of [0.5, 1.5) years is considered to catch annual monitoring. During the period of 1998-2004, we observe that around 50% of obligors change ratings in [0.5,1.5) years, which is consistent with the annual rating review policy of *Financing Company*. Some 10-13% of obligors' ratings are changed in time interval [0, 0.5) years. After 2004, we notice a significant drop in the time interval bucket of [0.5,1.5) years for the subsample groups lower than \$1000k, nearly 10% lower for sample A and 5% for sample B. This decrease is offset by the increase of obligor proportions in the time interval buckets of [0,0.5) and [1.5, 2.5] for subsample groups A and B. The fact of increasing proportion in the larger interval bucket [1.5, 2.5] could be explained either as a signal of stable credit quality of small loans, which keeps rating unchanged, or as the result of monitoring policy change in the *Financing Company*, which makes the review process less risk sensitive for small loans. The first explanation is hardly convincing in the context of the financial crisis during 2007-2009. Broadly speaking, the table indicates that the *Financing Company* dose not monitor each obligor with an equal time span (e.g., one year). In contrast with the high proportion (92.77%) of fixed reviewing time (12 months) in Mählmann (2006), the irregular pattern of observation times may make it more appropriate to apply the MT-Duration approach in this SME loan portfolio.

3.2.2 Characteristics of Non-rated Assignments

When an obligor discontinues, there is no subsequent record and we put it into a non-rated (NR) assignment at the end of the study period (e.g., one year). The reasons behind an NR rating event are not available. One possible case is that firms do not need to borrow or roll over their debt with another lender; an alternative case is that firms are not rated due to missing data (Gagliardini and Gourieoux, 2005)

It can be seen in Table 3.2 that the NR frequency tends to increase with the deterioration of credit risk quality, which corresponds to findings in many papers, e.g. Lando and Skodeberg (2002), Gagliardini and Gourieoux (2005). This fact could be explained as a signal of bad risk, which could create a selectivity bias. Moreover, Table 3.3 shows that NR frequencies decrease with loan size. For example, NR frequencies in subsample A with loan size [\$50k, 250k) are around 8-11% while those for sample C with loan size above \$1000k are around 3-8%. Meanwhile, we observe that NR frequencies increase with time until the period of 2008-2009. In our data, the NR rates range between 5% and 12%, which is not far from those of external rating systems. But there is evidence to show that the NR frequency is higher in internal rating systems than in external rating systems. For example, the NR frequency from the Banque de France used in Gagliardini and Gourieoux (2005) is rather higher (between 10% and 30%) than that from the S&P external rating system in Lando and Skodeberg (2002) (between 2% and 15%).

The technique which has emerged as an industry standard treats transitions to NR as non-informative. The effect of NR is therefore ignored by normalizing each transition frequency by the total fraction of obligors that do not have a NR rating. The Cohort method completely removes NR obligors from the sample; the intensity-based method considers NR obligors at least for the portion of the time for which they have a rating history before ending the period with NR assignments. Most recent studies use this proportional assignment approach to deal with the NR category based on two supporting arguments: (i) there were observations that few (13%) migrations to NR category are related to changes of credit quality, and that NR is not an indicator of immediate default (Carty, 1997; Foulcher et al., 2004) ; (ii) this approach makes results comparable across studies. But two issues emerge: the absence of selectivity bias and the magnified default probabilities for each rating. With high NR frequency as in the Gagliardini and Gourieoux (2005) case mentioned above, the removal of NR obligors could highly overestimate default probabilities. To deal with these issues, we propose an alternative approach based on the censored state technique of the MT-Duration method, which is demonstrated in a later section.

To conclude, the above analysis implies that the rating data for the *Financing Company* is incomplete in so far as the rating review policy depends on obligor-specific characteristics. Non-rated (NR) assignment may introduce bias in estimation of transition probabilities if it is treated as non-informative. Keeping these points in mind, we now compare and discuss the appropriate methods to accommodate the internal rating data.

3.3 The Methodologies for the Estimation of Migration Matrices

Credit migration risk is the transition probability of moving from one rating class to another within a given amount of time. In the context of a credit rating model, a set of K credit rating categories is denoted by a finite state space $S = \{1, 2 \ldots, K\}$, which is usually indexed with integers in the order of credit quality. State 1 represents the best credit rating, while last state K corresponds to "Default" which is an absorbing state. Once an obligor reaches the default state K, it is assumed to remain there forever. Transition probabilities are normally assembled into a matrix form called a transition or migration matrix. Conditional on an initial rating class, the $K \times K$ transition matrix $\mathbb{P}$ is a description of the possibilities of being in any of the various ratings classes after one period:

$$\mathbb{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1K} \\ p_{21} & p_{22} & \cdots & p_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ p_{K-1,1} & p_{K-1,2} & \cdots & p_{K-1,K} \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{3.3.1}$$

where $p_{ij} \geq 0$ $(i, j \in S)$. Since the elements in each row are mutually exclusive conditional probabilities, each row of the transition probability matrix must add up to 1; i.e., $p_{ii} \equiv 1 - \sum_{\substack{j=1 \\ j \neq i}}^{K} p_{ij}$ $(i, j \in S)$. A common practice in credit risk modeling is to ignore the possibility that an obligor recovers from the default state. Once an obligor reaches the default state K, it is assumed to remain there. Therefore we have $p_{KK} = 1$ and $p_{Kj} = 0$ $(j \neq K)$.

The rating grade for an obligor r at an arbitrary time t_n (n = 1, …, T) is denoted by $X^r(t_n)$, which has a finite number of possible values denoted by S. The time series behavior of $X = \{X^r(t_n) \mid n = 1, \ldots, T\}$ is governed by its conditional probability distribution, which is a function of the past rating history. Generally, we suppress the superscript when the definition is clear. Credit rating dynamics are commonly assumed to follow a first-order finite-state Markov Chain. The Markov property is an assumption on the conditional probability distribution that allows the future ratings of the obligor to rely only on the current rating, and to be independent of its past rating history. A finite state stochastic process with this property is called a first-order Markov chain (Cox and Miller, 1977). According to different time settings, we have a discrete-time Markov Chain (DTMC) and a continuous-time Markov Chain (CTMC).

The competing methods of transition probability estimation (i.e., Cohort, Duration and MT-Duration approaches) considered in this chapter have different time settings and data observation patterns. This section will describe the theoretical differences among the three competing methods. There are many introductory discussions of Markov Chain theory and related issues considered in this section; an appropriate reference is Noris (1998).

3.3.1 DTMC with Discrete Observations: Cohort Approach

In a discrete-time setting, the Markov property based on the conditional probabilities is

$$\begin{aligned} &Pr\big(X(t_{n+1}) = j \mid X(t_0) = i_0, X(t_1) = i_1, \ldots, X(t_n) = i\big) \\ &= Pr\big(X(t_{n+1}) = j \mid X(t_n) = i\big) \\ &\equiv p_{ij}(t_n, t_{n+1}). \end{aligned} \tag{3.3.2}$$

Here we identify the realization of X with its state.

Under the assumption of Markov chain behavior, the transition probability matrix over m-steps is easy to obtain by multiplication of each transition matrix over the steps. Industry practice often makes the assumption that the transition probabilities remain constant over time. If this assumption holds, a Markov chain is said to be time homogeneous. Then the transition probability matrix over m-steps is calculated by raising the one-period transition matrix to the power m:

$$\mathbb{P}(t_n, t_{n+m}) = \mathbb{P}(t_n, t_{n+1}) \times \mathbb{P}(t_{n+1}, t_{n+2}) \cdots \times \mathbb{P}(t_{n+m-1}, t_{n+m}) = \mathbb{P}^m,$$

where the m-step transition probability from i to j is the ij th element of $\mathbb{P}(t_n, t_{n+m})$ and each element of $\mathbb{P}$ is a constant value p_{ij}.

The conventional approach for estimating a transition probability matrix for a time-homogeneous DTMC with discrete time observations is the Cohort method. Considering equally-spaced discrete time $v_0, v_1, v_2, \ldots, v_m = T$, we let $\{x^r(v_n) \mid n = 0, 1, \ldots, m\}$ represent a realization of the Markov chain taken for obligor r from time v_0 to T, where each $x^r(v_n)$ is an element of S . Let P be the transition probability. Conditional on the distribution of obligors at v_0, the likelihood function of this realization of obligor r is (see

Commenges (2002)):

$$L_r(\mathbb{P}) = \prod_{n=0}^{m-1} P_{x^r(v_n)x^r(v_{n+1})}.$$

Assuming the independence of Markov chain process for each obligor, the full likelihood function is simply the product of individual likelihood contributions over all N obligors (see Mählmann (2006)).

$$L(\mathbb{P}) = \prod_{r=1}^{N} \prod_{n=0}^{m-1} P_{x^r(v_n)x^r(v_{n+1})}$$

According to the likelihood theory for DTMC in Guttorp (1995, Chapter 2), we define the transition counts N_{ij} as the number of migration from grade i to grade j over all obligors during the period and rewrite the likelihood in terms of the transition probabilities p_{ij} as defined in equation 3.3.2 assuming time homogeneity.

$$L(\mathbb{P}) = \prod_{i=1}^{K} \prod_{j=1}^{K} p_{ij}^{N_{ij}}.$$

The maximum likelihood estimator of the stationary transition probabilities can be derived imposing the K constrains $\sum_j p_{ij} = 1$; see Guttorp (1995, Chapter 2):

$$\hat{p}_{ij} = \frac{N_{ij}}{\sum_{k=1}^{K} N_{ik}}, i, j \in S. \tag{3.3.3}$$

The equation 3.3.3 provides the Cohort estimation of the elements of the transition matrix $\mathbb{P} = (p_{ij})$ between adjacent points during the period from 1 to T.

3.3.2 CTMC with Continuous Observations: Duration Approach

Lando and Skodeberg (2002) introduced a continuous-time Markov chain (CTMC) as an approach to the estimation of migration matrices. The time series behavior of CTMC is defined as a stochastic process $X = \{X(t) \mid t \geq 0\}$ which satisfies the following for all $t \geq 0$,

$s \geq 0$, and $i, j \in S$

$$Pr\big(X(s+t) = j \mid X(s) = i, \{X(u) : 0 \leq u < s\}\big)$$
$$= Pr\big(X(s+t) = j \mid X(s) = i\big)$$

With an analogy to a DTMC, time homogeneous CTMT satisfies the following:

$$\Pr\big(X(s+t) = j \mid X(s) = i\big) = Pr\big(X(t) = j \mid X(0) = i\big).$$

In discrete time, the time interval between transitions is a unit, regardless of the frequency of data. In continuous time, however, the time parameter t is continuous. In the context of credit rating migration, the time that an obligor spends in rating grade i before migrating from it follows an exponential distribution because of the Markov assumption (Noris, 1998). It means that there exists a positive constant rate q_i such that the waiting time of an obligor for leaving rating grade i is a random draw $t \sim exp(-q_i t)$, independent of its past rating history. Now let us introduce the transition intensity q_i, which is the probability that one transition occurs during a short interval, defined by

$$q_{ij} = \lim_{\Delta t \to 0} \frac{Pr\big(X(\Delta t) = j \mid X(0) = i\big)}{\Delta t},$$

assuming that the limit on the right-hand side exists. Then we construct the generator matrix $\mathbb{Q}$ as:

$$\mathbb{Q} = \begin{bmatrix} -q_1 & q_{1,2} & \cdots & q_{1,K-1} & q_{1,K} \\ q_{2,1} & -q_2 & \cdots & q_{2,K-1} & q_{2,K} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{K-1,1} & q_{K-1,2} & \cdots & -q_{K-1} & -q_{K-1,K} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.3.4}$$

where the entries of $\mathbb{Q}$ satisfy

$$
\begin{aligned}
\sum_{j=1}^{K} q_{ij} &= 0, \qquad \text{for } 1 \le i \le K \\
\sum_{i \neq j} q_{ij} &= q_i, \qquad \text{for } 1 \le i \le K \\
q_{ij} &\ge 0, \qquad \text{with } i \neq j.
\end{aligned}
$$

In Markov models with stationary transition intensities, the transition probabilities are linked to the transition intensities by the Kolmogorov differential equations $\frac{d}{dt}\mathbb{P}(t) = \mathbb{P}(t)\mathbb{Q}$ with unique solution subject to boundary condition $\mathbb{P}(0) = I$ as follows:

$$\mathbb{P}(t) = \exp(\mathbb{Q}t), t \ge 0. \tag{3.3.5}$$

Suppose that the process X^r is continuously observed from t_0 to T and we observe that the transitions have occurred at (exactly) times $t_0, t_1^r, t_2^r, \dots t_m^r \le T$ for each obligor r with a realization of state $\{x(t_n^r) \mid n = 0, 1, \dots, m\}$. Conditional on initial state of obligor at t_0, the likelihood function of this realization of obligor r can be expressed as (Commenges, 2002):

$$L_r(\mathbb{P}) = \prod_{n=0}^{m-1} \left[P_{x(t_n^r)x(t_n^r)}(t_n^r, t_{n+1}^r) q_{x(t_n^r)x(t_{n+1}^r)}(t_{n+1}^r)\right] P_{x(t_m^r)x(t_m^r)}(t_m^r, T)$$

Under the assumption of independent processes, we obtain the full likelihood for all obligors in terms of transition intensity q_{ij} by taking the equation 3.3.5 as (Inamura, 2006):

$$
\begin{aligned}
L(\mathbb{Q}) &= \prod_{r=1}^{N} \prod_{n=0}^{m-1} [\exp(-q_{x(t_n^r)}(t_{n+1}^r - t_n^r)) q_{x(t_n^r)x(t_{n+1}^r)}] \exp(-q_{x(t_m^r)}(T - t_m^r)) \\
&= \prod_{i=1}^{K} \prod_{j \neq i} q_{ij}^{N_{ij}(T)} \exp\big(-q_i R_i(T)\big),
\end{aligned}
$$

where $R_i(T) = \int_0^T 1_{\{x(s)=i\}} ds$ is the total time spent in rating grade i by time T. $N_{ij}(T)$ is the total number of transitions over the period from i to j where $i \neq j$. Then the maximum likelihood estimator for the elements of the generator matrix is given by

$$\hat{q}_{ij} = \frac{N_{ij}(T)}{R_i(T)} = \frac{N_{ij}(T)}{\sum_{r=1}^{N} t_i^r}, \tag{3.3.6}$$

where t_i^r is the total time spent in class i by each obligor r. One can estimate the transition probability matrix for the examined period by calculating equation 3.3.5 using the estimated generator matrix.

3.3.3 CTMC with Mixed Discrete-continuous Observations: MT-Duration Approach

Discussing the observation pattern in the context of likelihood inference for multi-state models, Commenges (2003) argued that the observations are always incomplete because we cannot make observations in continuous time but only at a finite number of distinct times. With fixed visit times, it is rare to witness the exact times of transitions during the follow-up period. As a result, the exact transition times are only known to occur during a particular interval, which is termed as interval-censoring. Joly et al. (2002) gave an example of the bias that occurs when one tries to treat interval-censored observation from an illness-death model as a continuous observation.

In reality, the most common pattern of observation is a mixing of discrete and continuous time observations. Taking the credit rating model as example, rating grades are observed in discrete time while the default can be observed at (nearly) exact time. It may happen that some status other than default are observed exactly. Moreover, in multi-states models for intermittently-observed processes, states may be censored as well as event times in some circumstances.

To develop notation under this incomplete observation scheme, the observations of $X^r = \{x(v_n^r) \mid n = 1, 2, \ldots, m\}$ are taken at discrete times $v_0, v_1^r, v_2^r, \ldots, v_m^r \leq T$ from v_0 to T for each obligor r. Let us call $\tilde{T}$ the follow-up time that is $\tilde{T} = min(D, T)$, where D is the time of default. We can observe $\tilde{T}$ and define the indicator function of default $\delta = I\{D \leq T\}$. For continuous intensities model, the individual likelihood function of this realization can be intuitively written as (Commenges, 2003):

$$L_r(\mathbb{P}) = \left[\prod_{n=0}^{m-1} P_{x(v_n^r)x(v_{n+1}^r)}(v_n^r, v_{n+1}^r) \right] \sum_{j \neq K} P_{x(v_m^r)j}(v_m^r, \tilde{T}) \Big(q_{jK}(\tilde{T}) \Big)^{\delta}. \tag{3.3.7}$$

In the context of credit rating models, if an individual obligor defaults (i.e., indicator $\delta = 1$), the rating grade on the previous instant before default is unknown. We call it in a censored state which belongs to the set $R = \{1, 2, \ldots, K - 1\}$. Then the contribution to the likelihood is summed over the unknown rating categories in set R on the time unit before default. If an individual obligor neither defaults nor has an observation at the end of study (i.e., indicator $\delta = 0$), it is in a censored state at time T and the censored state technique will be used at the end of study to contribute to the likelihood.

Equation 3.3.7 can be rewritten in terms of intensities $q_i j$ that are estimated as in the previous section using maximum likelihood. A closed form solution for these parameters in general does not exist and numerical procedures must be used to find the estimates. We will illustrate the estimation procedure in the following section.

3.3.4 An Example

To understand better the different estimation methodologies, we take an example from Lando and Skodeberg (2002) by way of illustration. Consider a rating system which consists of two non-default rating categories (A and B) and a default category D. We observe over one year the history of 20 obligors, 10 of which start in category A and 10 in category B. Assume that over the year, obligor 1 with A rating changes its rating to category B after one month and stays there the rest of the year. Assume that over the same period obligor 2 with B rating is upgraded after two months and remains in A for the rest of the period, and that obligor 3 which starts in B defaults after six months and stays there for the remaining part of the period.

Figure 3.1 demonstrates how obligors' credit quality might be monitored under intermittent observation in this hypothetical case. The dotted vertical lines represent the monitoring times (in months), and the solid horizontal arrows represent the true lengths of stay in each credit rating state. Take obligor 2 for example, this hypothetical B-rated obligor is observed to be in rating A at month 2. In reality, however, this obligor made an earlier transition from B to A at month 1. Similarly, B-rated obligor 3 is observed to default after 6 months. Actually, this obligor is upgraded to A one month before default.

Now we illustrate the three competing estimators of the one-year transition matrix.

The Cohort method takes a snapshot of the rating status at the beginning (month 0) and the end (month 12) of period (one year) to estimate up/down-grading entries of the transition matrix as follows:

$$\hat{p}_{AB}^{Coh} = \frac{N_{AB}}{N_{AA} + N_{AB} + N_{AD}} = \frac{1}{10} = 0.1;$$
$$\hat{p}_{BA}^{Coh} = \frac{N_{BA}}{N_{BA} + N_{BB} + N_{BD}} = \frac{1}{10} = 0.1;$$
$$\hat{p}_{BD}^{Coh} = \frac{N_{BD}}{N_{BA} + N_{BB} + N_{BD}} = \frac{1}{10} = 0.1.$$

Noting that the state D is assumed to be absorbing and that the diagonal elements are determined to make rows sum to one, we obtain the Cohort estimator of the one-year transition matrix

$$\mathbb{P}_{\widehat{Coh}(1)} = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The Duration method uses all the observations by assuming that the monitoring times are the exact times of rating grade transitions. The Duration estimator calculates up/downgrading transition intensity of generator $\mathbb{Q}$:

$$\hat{q}_{AB}^{Dur} = \frac{N_{AB}}{\sum_{r=1}^{20} t_A^r} = \frac{1}{9 + 1/12 + 10/12} = 0.1008$$
$$\hat{q}_{BA}^{Dur} = \frac{N_{BA}}{\sum_{r=1}^{20} t_B^r} = \frac{1}{8 + 2/12 + 6/12 + 11/12} = 0.1043$$
$$\hat{q}_{BD}^{Dur} = \frac{N_{BD}}{\sum_{r=1}^{20} t_B^r} = \frac{1}{8 + 2/12 + 6/12 + 11/12} = 0.1043$$

We obtain the estimated generator with the diagonal elements of the generator making rows sum to zero:

$$\mathbb{Q}_{\widehat{Dur}(1)} = \begin{bmatrix} -0.1008 & 0.1008 & 0 \\ 0.1043 & -0.2086 & 0.1043 \\ 0 & 0 & 0 \end{bmatrix}.$$

Accordingly, the Duration estimator of one-year transition matrix can be obtained using equation 3.3.5 as

$$\widehat{\mathbb{P}_{Dur}(1)} = \begin{bmatrix} 0.9086 & 0.0866 & 0.0048 \\ 0.0896 & 0.8161 & 0.0943 \\ 0 & 0 & 1 \end{bmatrix}.$$

The MT-Duration method The individual likelihoods in equation 3.3.7 can be written as follows:

$$\begin{aligned} L_1^{MT-Dur} &= p_{AB}(1/12)p_{BB}(11/12); \\ L_2^{MT-Dur} &= p_{BA}(2/12)p_{AA}(10/12); \\ L_3^{MT-Dur} &= p_{BA}(6/12)q_{AD} + p_{BB}(6/12)q_{BD}. \end{aligned}$$

It is useful to compare these components with the corresponding elements of the likelihood for the Duration approach:

$$\begin{aligned} L_1^{Dur} &= p_{AA}(1/12)q_{AB}p_{BB}(11/12); \\ L_2^{Dur} &= p_{BB}(2/12)q_{BA}p_{AA}(10/12); \\ L_3^{Dur} &= p_{BB}(6/12)q_{BD}. \end{aligned}$$

The difference of the role played by the intensity parameters is reflected in the interval-censored treatment of L_1 and L_2 in MT-Duration. Moreover, L_3 in MT-Duration reflects the censored state treatment of defaults.

The full likelihood of MT-Duration is accordingly a complicated function of the estimated intensities and must be estimated using numerical methods. These yield

$$\widehat{\mathbb{Q}_{MT-Dur}(1)} = \begin{bmatrix} -0.1129 & 0.1129 & 0 \\ 0.1178 & -0.2226 & 0.1048 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{\mathbb{P}_{MT-Dur}(1)} = \begin{bmatrix} 0.8989 & 0.0958 & 0.0053 \\ 0.0999 & 0.8060 & 0.0941 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.3.5 Methodological Comparisons

The Cohort method gives a snapshot of the rating states at the beginning and the end of required period. Multiple rating change activities occurring within the period are not captured. Accordingly indirect defaults via successive downgrade activities cannot be captured. This issue may lead to the fact of zero default probabilities in higher ratings due to the scarcity of default data. In the hypothetical example considered in the previous section, we have $\hat{p}_{AD}^{Coh} = 0$ since there is no observation of A-rated obligor default at the end of period. As well, it is often the case that observations are made at irregularly spaced time points, which may differ substantially across individual obligors. As we discussed in 3.2, the irregular pattern of observation time is found in our SME loan portfolio. Given such a reality, the cohort estimator, under the assumption of equally-spaced and identical times for all obligors, is not a appropriate estimator for transition probabilities. Conventionally, obligors whose ratings go to a non-rated (NR) assignment are treated as informative. They are removed from the Cohort estimates of transition matrices by normalizing each transition frequency by the total fraction of obligors that do not have a NR rating.

Unlike the Cohort method, the m-step transition probability in the Duration method incorporates more than one path of the rating process with successive migrations instead of the direct transition from state i at the beginning to state j at the end of period in the Cohort method. In reality, the major risk of high-credit quality asset lies in the possibility of downgrading with a subsequent increase in the likelihood of default. Accordingly, we can see positive estimator $\hat{p}_{AD}^{Dur} = 0.0048$ even though there is no obligor in the hypothetical case defaulting in one year from rating A. This indirect default effect in the Duration method also helps to capture more defaults for the highest-risk rating since most defaulted obligors are downgraded to a higher-risk rating before they actually default. It is worth noting that all the obligors that stay in a certain rating grade contribute to the information to estimate the transition intensity from this rating grade. In the hypothetical case, the 11-month stay in B of obligor 1 and 2-month stay in B of obligor 2 contribute to the q_{BD}, which counts a larger total time span for the rating transition and leads to lower probability of default from B in the Duration method ($\hat{p}_{BD}^{Dur} = 0.0943$) than that in the Cohort method ($\hat{p}_{BD}^{Coh} = 0.1$).

To illustrate how the time spent in a rating before a change impacts on the diagonal elements of migration matrices, we hypothesize two rating histories on these 20 obligors. Assume 19 obligors maintain a rating and that obligor 2 upgrades from B to A after 8 months on one scenario and after 11 months on another. The Cohort and Duration migration matrices for the two scenarios are shown in Table 3.4. We can see that the Cohort estimates give the same result regardless of transition time of obligor 2 with $p_{BB} = 0.9$; however, the longer the time obligor 2 stays in rating B, the higher the diagonal probabilities in the Duration approach: $p_{BB} = 0.9017$ for the case of 8 months stay and $p_{BB} = 0.9041$ for 11 months. This simple example provide an evidence that the Cohort method tends to overestimate the dynamic activities of a rating history in having higer estimates of the diagonal entries.

The MT-Duration approach accommodates 1) interval-censored transition times, 2) absence of information on the state occupied immediately prior to default, and 3) absence of information on the state occupied at the end of the follow-up period; that is, at the censoring date (if there is no assessment on this date). In the hypothetical case, for example, we only know that obligor 2 changed rating from B to A at time interval of (0,2] months. Assuming the exact time of rating change at month 2, the Duration method counts 2 months of time at p_{BB} for the likelihood of obligor 2 while the MT-Duration method counts 2 months of time at p_{BA}. Accordingly, the Duration method derives a larger time span for the diagonal elements of the intensity generator, which leads to a lower retaining intensity and higher diagonal probability. As we can see, $p_{AA}^{Dur}(= 0.9086) > p_{AA}^{MT-Dur}(= 0.8989)$ and $p_{BB}^{Dur}(= 0.8161) > p_{BB}^{MT-Dur}(= 0.8060)$.

In addition to accommodating interval-censoring time, the MT-Duration approach accounts for other limitations in the observation pattern. We notice that the rating occupied at the instant prior to default of obligor 3 is unknown. It might take path $B \to A \to D$ or path $B \to B \to D$. The MT-Duration method reflects this in the construction of the likelihood function which might lead to higher estimates of the probability of default compared to the Duration method: $P_{AD}^{MT-Dur}(= 0.0053) > P_{AD}^{Dur}(= 0.0048)$. Moreover, if an obligor is alive at the end of follow-up and does not have any monitoring on this date, then the state occupied at this time is also not known; we can only infer that the obligor is in a censored rating category at this time.

The hypothetical example, however, dose not involve this situation. In the *Financing Company*'s database, data is recorded in consecutive months without information concerning the rating review time. If an obligor doesn't default, it is assigned either a certain rating or a non-rated status at end of year. Naturally, the MT-Duration approach provides a methodology of censored state technique to deal with NR obligors.

As we discussed before, non-informative treatment of NR obligors increases credit migration as a whole. While inclusion of transitions history before NR in the Duration method captures more valuable information from NR obligors, the assumption of absence of selectivity bias still has to hold after non-rated events occur. Applying the methodology of MT-Duration, we will assign NR obligors to a censored state (which belongs to rating set $S = 1, 2, \ldots K$) instead of the NR category at the end of the examined period.

To summarize, under different assumptions of rating observation patterns, three competing methods provide different estimates of credit migration matrices. With less complexity of computation, the Cohort method is the most common approach in industry practice; even though its strong assumptions of equally-spaced and identical observation time may entail the loss of accuracy. The Duration method is increasingly adopted in the literature with the advantage of incorporating indirect migration. However, the continuous observation assumption of the Duration method throws some doubt on the estimates in the context of a mixed discrete-continuous observation pattern. Since the proposal of the MT-Duration method in Mählmann (2006), there hasn't been to our knowledge any application of this method.

In the context of our SME loan portfolio, we will examine the differences among these three competing methods.

3.4 Empirical Results

Using internal rating data from the *Financing Company*, annual migration matrices from 1998 to 2009 for the four sample groups are estimated using the three competing estimation approaches presented in the previous section. Mählmann (2006) used average one-year horizon estimation over 6 years (1996-2002) to compare the results between the Cohort and the MT-Duration methods. The advantages of taking annual estimates in our research are two fold: the assumption of time-homogeneity is relaxed somewhat, and computational complexity of MT-Duration is considerably reduced.

In this section, we will also examine the differences among the estimated transition matrices using different distance metrics.

3.4.1 Transition Matrices

NBER defined the years 2001 and 2007-2009 as recession periods for the U.S. The time series of default rates in our data also exhibits high points in the periods 2001-2002[1] and 2008, and a low point in 2006. Therefore we select three years — 2001, 2006 and 2008— to present transition matrices estimated by the three competing methods: they are illustrated in Tables 3.5 - 3.7 for the total sample. These matrices are given for illustrative purposes. To compare these results among methods for the three years and the four sample groups, we focus on the diagonal probabilities of no rating change, probabilities of default and up- and down-grade probabilities. The results are summarized respectively Tables 3.8 - 3.11 and Figures 3.2 - 3.5. The next subsection discusses in greater detail the different results.

3.4.1.1 Comparison of Diagonal Probabilities

The diagonal elements of the migration matrix give the probability of an obligor retaining its current rating. In general, a migration matrix shows a high probability load on the diagonal. For example, in the S&P average annual transition matrix, the diagonal probabilities were between 70-90% (Poor's, 2013). Those from the internal rating system of JPMorgan Chase's wholesale portfolio were between 60-90% (Araten et al., 2004). In contrast, results for our SME loan portfolio exhibit considerably smaller probabilities of no rating change. Using the Cohort method (see Table 3.8), diagonal elements (no rating change) are between 30-60% in 2001, 50-70% in 2006 and 60-80% in 2008. These results are comparable to those of a

[1]The high point of PDs is 2002 for sub-sample A and B, but 2001 for sub-sample C. We select 2001 to accommodate the bootstrap experiment in the next section.

German bank's loans portfolio with 85% SME obligors, which provided 40-70% diagonal probabilities (Mählmann, 2006). As we discussed in Chapter 2, the SME's characteristic of volatile earnings and growth has a consequence that fewer SME obligors could maintain the same credit quality for a given period compared to commercial loan obligors.

The rating review policy of *Financing Company* has grown less risk-sensitive for obligors with small loans (see Section 3.2). The monitoring time for loans below $1000k was extended to more than one year after 2004. Due to this policy change, we observe in Table 3.8 that diagonal probabilities in subsamples A and B monotonically increase along the three years. Moreover, we found that the average difference [2] of diagonal probabilities between the Cohort method and the Duration method are 28%, 9% and 3% for the three years respectively in subsample A. The policy change leads to less rating movement within one year, which reduces the differences among the three methods. This observation reflects that the difference between the Cohort method and intensity-based method should not be ignored if the financial institution has frequent rating monitoring policy.

To avoid the effect of policy change on the estimation, we take subsample C to examine the differences among three methods. As shown in Figure 3.2, the Cohort method provides much lower diagonal probabilities of no rating change than the two intensity-based methods. The Duration method estimates the highest results which are fairly close to those of the MT-Duration method. In year 2001, for example, diagonal probabilities using the Cohort, the Duration and the MT-Duration methods are between 30-54%, 42-61% and 40-60% respectively (see Table 3.8). These empirical results echo the theoretical analysis of the three competing methods in the previous section. If obligors are not under equally spaced and identical time of credit monitoring, the Cohort method may overestimate the extent of credit migration; if obligors are not reviewed continuously, the Duration method may overestimate rating immobility.

The difference [3] between the Cohort and the MT-Duration methods are considerably higher (up to 32%) than the result (lower than 3%) in Mählmann (2006).

3.4.1.2 Comparison of Default Probabilities

In the last column of the migration matrix, we obtain the probabilities of default (PDs) for each risk rating. It provides an estimate of the likelihood that an obligor will be unable to

[2] The difference is to divide difference of estimation between the Cohort and Duration methods by the Duration estimation.

[3] The difference is to divide the difference of estimation between the Cohort and MT-Duration methods by the MT-Duration estimations.

meet its debt obligations. As we can see in Table 3.9, PDs monotonically increase with the deterioration of credit quality in all sample groups for each year and estimation method. The counter-cyclical characteristic of PDs as discussed in Chapter 2 is demonstrated here as well. In each subsample group, we always observe the high PDs in the recession years 2001 and 2008. During the severe financial crisis in 2008, the PDs are much higher than those in 2001.

Looking within the three subsamples, we see that subsample C with large loans above \$1000k exhibits a higher risk of default although obligors in this subsample have lower risk ratings than the other two subsamples. As shown in Figure 3.3, PDs for subsamples A and C are higher than those for subsample B. Specifically, subsample C has higher PDs for high-risk ratings, while subsample A holds higher PDs for low-risk ratings. Holding 56% of the total portfolio exposure, subsample group C could bring significant potential loss with such high PDs.

Except for the lowest- and highest-risk rating, we notice that the Cohort estimation provides the highest PDs and the Duration method estimates the lowest ones. With regard to the total sample, we see that the Cohort method estimates 0.2-2.5% more PDs than the Duration method ; while the MT-Duration method estimates 0.1-0.8% more PDs. Low intensity-based probability estimates could be obtained if an obligor spends time in a non-default rating grade before the default. This would reduce the default intensity and thereby reduces PDs. The MT-Duration method accounts for more possible defaults than the Duration approach by using the censored-state technique. This would amplify the downgrade effect so as to raise PDs.

3.4.1.3 Comparison of Downgrade and Upgrade Probabilities

The cells to the right (left) side of the diagonal of a migration matrix represent down(up)-ward rating migration movements. We summarize up down(up)-grade probabilities for each rating category in Tables 3.10 - 3.11 and Figures 3.4- 3.5. Since the row of a migration matrix sums to 1, we observe the following down/upgrade behavior: (i) Looking into three subsamples, we see that down- and up-grade probabilities increase with loan size in 2006 and 2008. (ii) Along the three selected years (2001, 2006 and 2008), down- and up-grade probabilities for subsamples A and B decrease. Subsample C exhibits a cyclical pattern: down(up)grade probabilities increase (decrease) in adverse environments, and vice versa. (iii) The differences among three competing methods decrease along the selected years for subsamples A and B.

Comparing the three competing methods, Figure 3.4 indicates first that the Cohort method estimates the highest down/upgrade probabilities and the Duration provides the lowest ones. Second, the differences between the Cohort and intensity-based methods are larger than that between the Duration and the MT-Duration methods. Consider the total sample in 2001 for example: the downgrade probabilities for rating 3 from the Cohort, Duration, and MT-duration methods are 47%, 37% and 39% respectively; the upgrade probabilities are 25%, 18% and 19%.

3.4.2 Formal Comparisons Between Transition Matrices

To answer the question whether the three methods produce transition matrices that are statistically differentiable, one needs a metric to measure a scalar difference between two matrices. Traditional matrix norms are classical cell-by-cell distance measures. Though these intuitive measures were widely used to compare migration matrices in the early research literature (Belkin et al., 1998; Israel et al., 2001; Bangia et al., 2002; Wei, 2003). However, they are not appropriate to measure differences in migration matrices (Trueck and Rachev, 2009, chap.7). The mobility-based metric proposed by Jafry and Schuermann (2004) and risk-sensitive difference indices introduced by Trueck and Rachev (2005) are used most often in the current literature (Lando and Skodeberg, 2002; Xing et al., 2010; Trück, 2008; Berteloot et al., 2013). In this section, we use these two metrics to examine the differences among the three migration matrices estimated by the Cohort, Duration and MT-Duration methods.

The mobility-based metric proposed by Jafry and Schuermann (2004) associates a scalar to capture the overall dynamic features of a given transition matrix. The central characteristics of a migration matrix is the amount of migration (or "mobility") imposed on the state vector from one period to the next. A so-called mobility matrix $\tilde{P}$ is constructed from the original matrix $\mathbb{P}$: $\tilde{\mathbb{P}} = \mathbb{P} - \mathbb{I}$. A mobility matrix is only the dynamic part of the original matrix and reflects the "magnitude" of $\mathbb{P}$ in terms of the implied mobility, in so far as the diagonal is the negative values of the sum of the row element. A metric is based on the manipulations of $\tilde{\mathbb{P}}$ which satisfies some performance criteria with respect to the original matrix $\mathbb{P}$. Following previous studies (Geweke et al., 1986) and the definition of a matrix's norm in Strang (1988), Jafry and Schuermann (2004) proposed the average of all the singular values of a matrix to capture the general characteristic of a matrix. This mobility-based metric

(SVD) is defined as:

$$M_{SVD}(\mathbb{P}) = \frac{\sum_{i=1}^{n} \sqrt{\lambda_i(\tilde{\mathbb{P}}'\tilde{\mathbb{P}})}}{n}. \tag{3.4.1}$$

There is a natural interpretation of SVD. Suppose that the value of the off diagonal elements of a transition matrix $\mathbb{P}$ of order N for each row is p. Moreover, assume that the value of each off diagonal element is $p/(N-1)$. Then $M_{SVD}(\mathbb{P}) = p$ (See details in Jafry and Schuermann (2004)). Differences in the value of this metric corresponds to deviations from the "average" transition matrix just deferred. . In this context, the difference between two migration matrices could be measured by M_{SVD}:

$$D_{SVD}(\mathbb{P}_1, \mathbb{P}_2) = M_{SVD}(\mathbb{P}_1) - M_{SVD}(\mathbb{P}_2), \tag{3.4.2}$$

which gives a directional deviation between two matrices in terms of the mobility or approximate averge probability of migration.

In another approach, the risk-sensitive difference indices introduced by Trueck and Rachev (2005) measures the difference between two migration matrices $\mathbb{P}_1$ and $\mathbb{P}_2$ by weighted cell-by-cell calculation: the weight of a row element is based on the distance of the element from the diagonal of matrix and whether it is on the left- or right- side of the diagonal. The weight captures the difference between near and far migrations, and incorporates the direction of the shift. Following Trueck and Rachev (2005), we use the weight for a cell (i, j) between matrices $\mathbb{P}_1$ and $\mathbb{P}_2$ defined as:

$$d(i, j) = (i - j)(p_{ij} - q_{ij}).$$

Accordingly, large transitions get a higher weight than near transitions. The sign of term $(i-j)$ captures a risk dimension. As well, in so far as default events have more direct impact, differences in the default column obtain different weights than the differences of other cells. We then define two risk-sensitive indices following Trück (2008):

$$\begin{aligned} D1(\mathbb{P}_1, \mathbb{P}_2) &= \sum_{i=1}^{n}\sum_{j=1}^{n-1} d(i,j) + \sum_{i=1}^{n} n\cdot d(i,n), && (3.4.3)\\ D2(\mathbb{P}_2, \mathbb{P}_2) &= \sum_{i=1}^{n}\sum_{j=1}^{n-1} d(i,j) + \sum_{i=1}^{n} n^2\cdot d(i,n) && (3.4.4)\end{aligned}$$

where n is the dimension of the matrix.

3.4.2.1 SVD Metric Comparison

We use M_{SVD} to compare the mobility "size" of the migration matrices across the different estimation methods for the period 1998 - 2009; the values are shown in Figure 3.6. We can clearly see that the Cohort matrices are always "larger" than the matrices estimated by the other two intensity-based methods. The differences between two intensity-based methods are relatively small with the MT-Duration matrices "larger" than the Duration matrices.

Since the difference in M_{SVD} correspond to the deviation from the average mobility magnitude, lower M_{SVD} indicates less migration and higher probabilities of staying in the diagonal. The graph reflects the general result that the Cohort approach leads to lower values on the diagonal than do the two intensity-based approaches. The relatively high values of MT-Duration relative to Duration reflects the mixed nature of the estimation method.

Looking into the sub-samples, we find different patterns of matrices mobility. The M_{SVD} values in subsamples A and B dramatically decrease after the year 2004. It corresponds to the fact of increasing values of the diagonal elements of transition matrices for these two subsamples after the year 2004; this was highlighted in Section 3.4.1. The rating review policy change in the *Financing Company* for loans below $1000k leads to longer time span between successive rating monitoring which causes higher retaining rate and lower mobility within the transition matrices.

To test whether the matrices are statistically different, the distributional properties of ΔM_{SVD} are obtained through a non-parametric bootstrap experiment as in Hanson and Schuermann (2006). By resampling on the obligor rating history randomly, we create N bootstrap samples of size B_t, where B_t is the number of obligor-histories over some time interval which could be a year or multiple years. The nonparametric bootstrap based on resampling presumes that the data is independent and comes from the same distribution. It is difficult to assume independence across multiple years because of the mobility of the transition matrix over the business cycle, but is more appropriate at a shorter horizon such as one year; this approach is typical for many risk management applications.

Because of unstable rating policy for small loans, we use only subsample C to conduct the bootstrapping experiment. As above, we focus on years 2001, 2006 and 2008. With $N = 1000$ replications, we find that the difference between the Cohort and the Duration methods are statistically significant; however, the difference between the Duration and MT-Duration methods are also statistically significant as well even though the difference appears small as shown in Figure 3.6.

3.4.2.2 Risk-sensitive Metric Comparison

We use the risk-sensitive metric $D1$ to measure the difference between the Cohort and the Duration migration matrices $D1(\mathbb{P}_{Coh}, \mathbb{P}_{Dur})$ as well as the difference between the MT-Duration and the Duration migration matrices $D1(\mathbb{P}_{MT-Dur}, \mathbb{P}_{Dur})$ for the period 1998-2009. The results for four samples are shown in Figure 3.7. We see that the differences between the Cohort and the Duration migration matrices are larger than those between the two intensity-based migration matrices. Since metric D1 assigns negative signs for the elements on the right side of diagonal probabilities, the figure reflects that the Cohort migration matrices estimate larger downgrading risk than the Duration migrations do; the MT-Duration migration matrices produce more risk than the Duration estimates as well.

Since the results using metric $D2$ are similar to those of $D1$ as above, we do not present them here. They are available on request.

3.5 Economic Relevance: Credit VaR

The differences between the competing transition matrices estimates reviewed by the metrics may not have considered the economic relevance. To illustrate, we look at credit VaR (value-at-risk) implied by the credit portfolio models which are used to generate the loss distribution of a credit assets portfolio.

3.5.1 Simulation Methodology

The purpose of credit VaR for financial institutions is to calculate economic capital that provides a cushion against potential losses. There are a variety of models that can be used to calculate economic capital for a given portfolio of credit assets. One popular credit portfolio model, CreditMetrics®, computes the portfolio capital value via an ordered probit model including not only defaults but also credit migrations. To capture the migration revaluation of assets, CreditMetrics needs the assets returns as inputs; however, they are unavailable in our study. Therefore, as in Gordy (2000), we take a restricted version of CreditMetrics, which is a model of only default risk losses. We restrict the set of outcomes to two states, default and non-default. In the event of default, the loss is a fixed fraction λ of the exposure at default (EAD). This is a second significant simplification of the full CreditMetrics implementation, which allows idiosyncratic risk in recoveries. In the non-default state, the loan retains its book value.

The latent variable X_r for obligor r is driven by a systematic risk factor Z and an idiosyncratic noise component ϵ_r:

$$X_r = w_r \mathrm{Z} + \eta_r \epsilon_r; \tag{3.5.1}$$

The vector of factor loadings w_r determines the relative sensitivity of obligor r to the risk factors, and the weight η_r determines the relative importance of idiosyncratic risk for the obligor. The Z is assumed to be normally distributed with mean zero and covariance Ω. ϵ_r are assumed to be identical independent random variables that follow a standardized Gaussian distribution. It is assumed that X_r has variance 1 (i.e., that $w_k' \Omega w_k + \eta_k^2 = 1$). Under the assumption of normality, the default thresholds C_i associated with each rating grade i are obtained by $C_i = \Phi^{-1}(p_{iK})$, where p_{ik} is the default probability. When the latent variable X_r falls under the threshold C_i, the obligor defaults.

The model of losses is estimated by Monte Carlo simulation. For simplicity, we assume a single systemic risk factor Z. All the obligors have the same weight w on the systematic risk factor and η (i.e., $\sqrt{1-w^2}$) on the idiosyncratic risk factor. To obtain a single trial on the portfolio, we randomly draw a single Z and a set of iid $N(0,1)$ idiosyncratic ϵ_r. We form the latent X_r for each obligor, which is compared against the threshold values C_i to determine the default indicator variable D_r (one for default, zero otherwise). The portfolio loss for this trial is given by $\sum_r (EAD_r \cdot \lambda \cdot D_r)$. To estimate a distribution of portfolio outcomes, we repeat this process many times. The portfolio losses for each trial are sorted to form a cumulative distribution. The VaR (Value-at-Risk) at a specific confidence level can be obtained by taking the appropriate quantile of this loss distribution.

3.5.2 Simulation Results

We consider as in previous section, several years are of particular interest: 2001, 2006 and 2008. This section will present the simulation results for each of three years for the three competing estimation methods.

The rating distribution of portfolio for simulation is displayed in Table 3.12. For each simulated year, there are four portfolios associated with the sample groups. The outstanding loan amount is used as EAD (exposure at default). A 45% recovery rate and 0.21 factor loading (w^2) are assumed.

With 100,000 simulations, we summarize our findings in Table 3.13 - 3.15 which displays the mean, standard deviation of portfolio loss and VaR (Value-at-Risk) figures at 99% and 99.9% for the different rating distributions. Most of the ratios of credit VaR (cohort to duration) are greater than one with the largest deviation up to 14%, while the ratios of credit VaR (MT-Duration to duration) deviate up to 9%. This conveys two points. First, the Credit VaR implied by the cohort method is typically highest, then followed by the MT-Duration method, and Duration. In this portfolio, the Cohort method tends to overestimate default risk while Duration method tends to underestimate default risk. MT-Duration method captures the potential loss information of downgrade and defaults to provide the medium default probabilities between the Cohort and the Duration methods. Recall that the Cohort method provides lower default probabilities for the extreme rating grades than the intensity-based methods. However, the observation here is an overall effect on the default risk of a portfolio. Second, without exception, the differences between the Cohort and the more efficient intensity-based methods (Duration and MT-Duration) are larger than that between the intensity-based methods for each year.

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3.6 Markovian Behavior and Relaxation of Homogeneity

Each of the three methods considered so far for the rating migration process makes two specific assumptions: we assume first-order Markov property and time homogeneity. In this section, we will examine the Markovian behavior of our SME rating changes and investigate the impact of the time-homogeneity assumption on the one-year time horizon migration matrices.

The Markovian behavior of credit rating dynamics is examined via the construction of Up-, Maintain- and Down-Momentum matrices (Bangia et al., 2002) to see whether an upgrade (downgrade) is more likely followed by an upgrade (downgrade). We note in passing that the Markovian property could be examined using the transition intensity via a proportional hazard model; the results of this approach is similar to the one obtained here and are available on request.

A method based on continuous time setting, the Aalen-Johansen estimator, is used to calculate the migration matrices in a way that relaxes the time-homogeneous assumption. The estimates obtained are then compared with time-homogeneous estimates of migration matrices based on the Duration and MT-Duration methods.

As discussed in previous sections, to avoid the impact of rating policy changes, we use only subsample C with loan sizes above \$1000k to investigate the properties of migration matrices. Each analysis is done for the selected three years 2001, 2006 and 2008.

3.6.1 Markovian Behavior

In research conducted by Bangia et al. (2002) and Lando and Skodeberg (2002) using external rating data, the first-order Markov property has been rejected by testing for rating drift. Two-period changes like "Down-Down" or "Up-Up" are generally considered to be more probable than alternative rating changes like "Down-Up" or "Up-Down". These results violate the first-order Markov property.

In order to investigate whether such a rating drift exists in our data, we apply the same procedure as in Bangia et al. (2002). The cohort of obligors each year is separated into three subgroups according to their rating experience in the previous year: upward, downward and no change. More specifically, the matrix M includes the total number of transitions from one rating grade to another during $t-1$ to t, in which $\{M(t)\}_{ij}$ gives the number of transitions from rating grade i at time $t-1$ to rating grade j at time t. The matrix M is split into the sum of three matrices, called Up-Momentum-Matrix, Maintain-Momentum-Matrix, and Down-Momentum-Matrix. These three matrices are defined element-by-element in the following way:

$$\begin{aligned}
\{M_{Up}(t)\}_{ij} &= \text{number of transition during } t-1 \text{ to } t \text{ from } i \text{ to } j \text{ of obligors that were} \\
&\quad \text{upgraded during the year } t-2 \text{ to } t-1 \\
\{M_{Maintain}(t)\}_{ij} &= \text{number of transition during } t-1 \text{ to } t \text{ from } i \text{ to } j \text{ of obligors that have} \\
&\quad \text{no rating change during the year } t-2 \text{ to } t-1 \\
\{M_{Down}(t)\}_{ij} &= \text{number of transition during } t-1 \text{ to } t \text{ from } i \text{ to } j \text{ of obligors that are} \\
&\quad \text{downgraded during the year } t-2 \text{ to } t-1.
\end{aligned}$$

By construction, we have

$$M(t) = M_{Up}(t) + M_{Maintain}(t) + M_{Down}(t).$$

We present the results for the years 2001, 2006 and 2008 in Table 3.16-3.18. The transition probability p_{ij}^{Up} for the Up-Momentum-Matrix is obtained as the ratio of transition number of obligors from i to j ($\{M_{Up}(t)\}_{ij}$) to the total transition number of obligors with initial rating i ($\sum_{j=1}^{K} M_{Up}(t)_{ij}$). Similarly, we could calculate transition probabilities for Maintain-Momentum-Matrix ($M_{Maintain}(t)$), Down-Momentum-Matrix($M_{Down}(t)$) and Unconditional Matrix ($M(t)$). Then we follow the literature to use two methods to examine the path dependence of migration matrices.

First, following Krüger et al. (2005), we check the relation between the upgrade probabilities and downgrade probabilities for each momentum-based migration matrix. For each rating in a momentum matrix, we take the sum of all elements to the right side of the diagonal as downgrade probabilities and sum of all elements to the left side of the diagonal as upgrade probabilities. The results for the years 2001, 2006 and 2008 are shown in Table 3.19. Take rating 3 as an example: if an obligor has experienced a downgrade the year before, it has a 41% chance of an upgrade and 18% chance of a downgrade. Similarly, if an obligor has been upgraded (maintained) the year before, the chance of upgrade vs. downgrade is 25% vs. 34% (28% vs. 40%).

Previous studies have found a so-called rating drift where the probabilities of an upgrade (downgrade) is increased if the previous period saw an upgrade (downgrade). The *Financing Company*'s rating data does not appear to have rating drift (except for middle ratings in Up-Momentum-Matrix in 2006). The non-Markov feature in this data results in rating reversal rather than rating drift.

A second approach follows the analysis in Bangia et al. (2002). Here we compare the differences of up- and down-grade probabilities between a momentum matrix and the unconditional matrix $M(t)$ (migration matrix without considering its previous path of rating change) as in Table 3.20. Here look at rating 3 in 2006, the entry 1.2 in Down-Momentum columns is equal to (15.89 + 12.15 + 17.76) minus (15.86 + 12.63 + 16.13). According to the Markov property, we would expect the entries in this table around zero. We see that the downgrade probabilities for most ratings from the Down-Momentum-Matrix are smaller than the corresponding values in the unconditional matrix. The opposite is true for the Up-Momentum-Matrix. Again, we have evidence of rating bounce. This result is contrary to the studies in Bangia et al. (2002), which observed rating drift using external rating data.

However, in line with studies in Bangia et al. (2002), the average default probability of SME obligor is more sensitive to a prior downgrading history. Take 2008 as an example, in Table 3.18, the down-momentum average default rate (10.6%) is 40% larger than the unconditional one (7.5%), whereas the up-momentum average default rate (5.1%) is 30% lower than the unconditional expectation. Similarly, Bangia et al. (2002) found that the down-momentum average default rate is nearly five times as large as the unconditional one, whereas the up-momentum average default rate is less than one fifth of the unconditional expectation.

Summarizing the results of this subsection we do not observe rating drift, i.e., two period changes like "Down-Down" or "Up-Up", which is in contrast to the rating drift result based on external rating data (Altman and Kao, 1992; Bangia et al., 2002). Interestingly, the *Financing Company*'s obligors tend to compensate for previous-period rating changes. The obligors in a certain rating category which have been downgraded in the previous period are more likely to be upgraded in the next period and vice versa. This finding is similar to the studies with the internal rating data in Krüger et al. (2005) (Deutsche Bundesbank) and Mählmann (2006) (a medium-sized German bank).

3.6.2 Relaxing Time-homogeneity

Time-homogeneity is an extreme assumption over the long run. However, some studies (Lando and Skodeberg, 2002) show that there is only a marginal difference between estimates of the Duration migration matrices with and without the assumption of time-homogeneity for a one-year time horizon. Here we would like to examine the estimates of SME migration matrices without assuming time-homogeneity.

Following Lando and Skodeberg (2002), we use the Aalen-Johansen estimator to examine the non-homogeneous case. Consider a continuous time, non-homogeneous, Markov process with finite state space $S = \{1, 2, \ldots, K\}$. Its migration matrix for the period from s to t is denoted by $\mathbb{P}(s,t)$. Given that a sample has m transitions over the period from s to t. The paper shows that $\mathbb{P}(s,t)$ could be estimated consistently by

$$\hat{\mathbb{P}}(s,t) = \prod_{i=1}^{m} (I + \Delta \hat{A}(T_h)), \tag{3.6.1}$$

where T_h is a jump time in the period from time s to t corresponding to rating changes and

$$\Delta \hat{A}(T_i) = \begin{bmatrix} -\frac{\Delta N_1(T_i)}{Y_1(T_i)} & \frac{\Delta N_{12}(T_i)}{Y_1(T_i)} & \frac{\Delta N_{13}(T_i)}{Y_1(T_i)} & \cdots & \frac{\Delta N_{1k}(T_i)}{Y_1(T_i)} \\ \frac{\Delta N_{21}(T_i)}{Y_2(T_i)} & -\frac{\Delta N_2(T_i)}{Y_2(T_i)} & \frac{\Delta N_{23}(T_i)}{Y_2(T_i)} & \cdots & \frac{\Delta N_{2k}(T_i)}{Y_2(T_i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\Delta N_{k-1,1}(T_i)}{Y_{k-1}(T_i)} & \frac{\Delta N_{k-1,2}(T_i)}{Y_{k-1}(T_i)} & \cdots & -\frac{\Delta N_{k-1}(T_i)}{Y_{k-1}(T_i)} & \frac{\Delta N_{k-1,k}(T_i)}{Y_{k-1}(T_i)} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

$\Delta N_{ij}(T_h)$ is the transition number observed from state i to j at time T_h. $\Delta N_k(T_h)$ is the total transition numbers away from intial state k and $Y_k(T_h)$ counts the obligors in state k before time T_h. Note that the row of matrix $I + \Delta \hat{A}(T_h)$ sums to 1. We could view the Aalen-Johansen estimator as the Cohort method used in an extremely short time interval.

Using the example in section 3.3.4, we illustrate how to produce a one-year Aalen-Johansen estimator of the migration matrix. First, we calculate $\Delta A(T_h)$ as:

$$\Delta A(T_{1/12}) = \begin{bmatrix} -\frac{1}{10} & \frac{1}{10} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Delta A(T_{2/12}) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{11} & -\frac{1}{11} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Delta A(T_{6/12}) = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{10} & \frac{1}{10} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we could obtain the Aalen-Johansen estimator:

$$\hat{\mathbb{P}}(0,1) = \begin{bmatrix} 0.9091 & 0.0818 & 0.0091 \\ 0.0909 & 0.81818 & 0.0909 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can see that there is a difference between the Aalen-Johansen estimator and the Duration estimates, see section 3.3.4. Lando and Skodeberg (2002) argues that the matrix exponential of the Duration method could be viewed as a smooth version of the Aalen-Johansen estimator.

Tables 3.21 - 3.23 show the comparison of one-year migration matrices estimated based on the Aalen-Johansen estimator, the Duration and the MT-Duration methods for the selected years 2001, 2006 and 2008. We see that there are no dramatic differences between the Aanlen-Johansen estimates and the Duration (or MT-Duration) for our large data sets, which is similar to the results in Lando and Skodeberg (2002) based on external rating data. The difference is less profound than the one between the Cohort method and the intensity-based methods. This fact supportss the industry practice of maintaining one-year PD for each rating through time and taking a one-year horizon for credit risk assessment.

3.7 Conclusions

Encouraged by Basel II, the internal-ratings based (IRB) approach has been increasingly adapted by financial institutions and banks to determine capital requirements. As key inputs, accurately estimated transition probability matrices associated with internal ratings have become important. Using the internal rating data of the SME loan portfolio from a Canadian *Financing Company*, this paper empirically compares the three competing methods (i.e., Cohort, Duration and MT-Duration) based on different assumptions regarding the observation pattern of transitions.

First, we investigated the features of internal rating data of the SME loan portfolio from the dual dimensions of rating and loan size. In line with previous studies, our data exhibits incomplete observations, along with varying time intervals between rating assessments, and a high percentage of non-rated (NR) obligors. Those particularities of the internal rating data motivated us to compare three multi-state Markov chain methods that have been used in industry and academic research.

An example is introduced to illustrate the three competing methods. With the assumption of discrete observation, equally spaced and identical times for all obligors, the Cohort method ignores multiple rating change activities withing the studied period and can not accommodate the irregular observation pattern of internal rating data. By contrast, the Duration method performs the estimation with the assumption of continuous observations, which could capture the indirect default effect. However, the most common pattern of observation is, in reality, a mixed discrete-continuous time observation. With such assumption, the MT-Duration approach is able to accommodate (1) interval-censored transition times, (2) absence of information on the state occupied immediately prior to a default, and (3) absence of information on the state occupied at the end of follow-up period (which we apply on the non-rated obligors).

Estimated migration matrices for each year are produced by the three competing methods from 1998 to 2009. We compare these results for four samples and the selected three years - 2001, 2006 and 2008 - that are associated with the business cycle. This cell-by-cell analysis focuses on the diagonal, default and up/downgrade probabilities. We find that the estimates of differences between the Cohort and the two-intensity based estimates are larger than those between the Duration and the MT-Duration methods. This result is observed again when we compare the differences of migration matrices using mobility-based and risk-sensitive metrics.

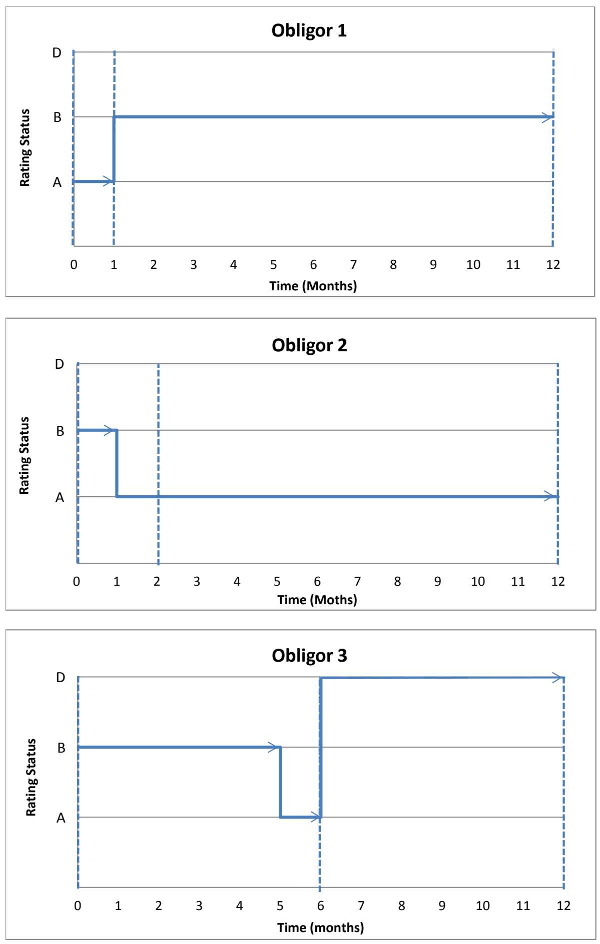
Economic relevance is analyzed via the simulation of credit VaR implied by the one-factor CreditMetric model. Indeed, we find that the estimation method matters for one-year migration matrices estimation. The Cohort method tends to overestimate the default risk, while the Duration method underestimates it. Capturing missing information, MT-Duration provides probabilities of default between the Cohort and Duration estimates. Credit VaR differences between the Cohort and the more efficient intensity-based methods are larger than between the Duration and MT-Duration methods, with the largest difference being 14% for the former, and 9% for the latter.

In addition, the assumptions behind the three methods, i.e., time-homogeneous first-order Markov chain, are examined for our SME rating migration process. We find evidence for SMEs' rating reversal activity contrary to the rating reversal aversion seen in the external rating data. However, the default probabilities of SME obligors are more sensitive to a prior downgrading history.

Using the Aalen-Johansen estimator to estimate migration matrices for inhomogeneous Markov chain, we observe that there is only a marginal difference between the continuous-time migration matrices estimates with and without the assumption of time-homogeneity for a one-year time horizon.

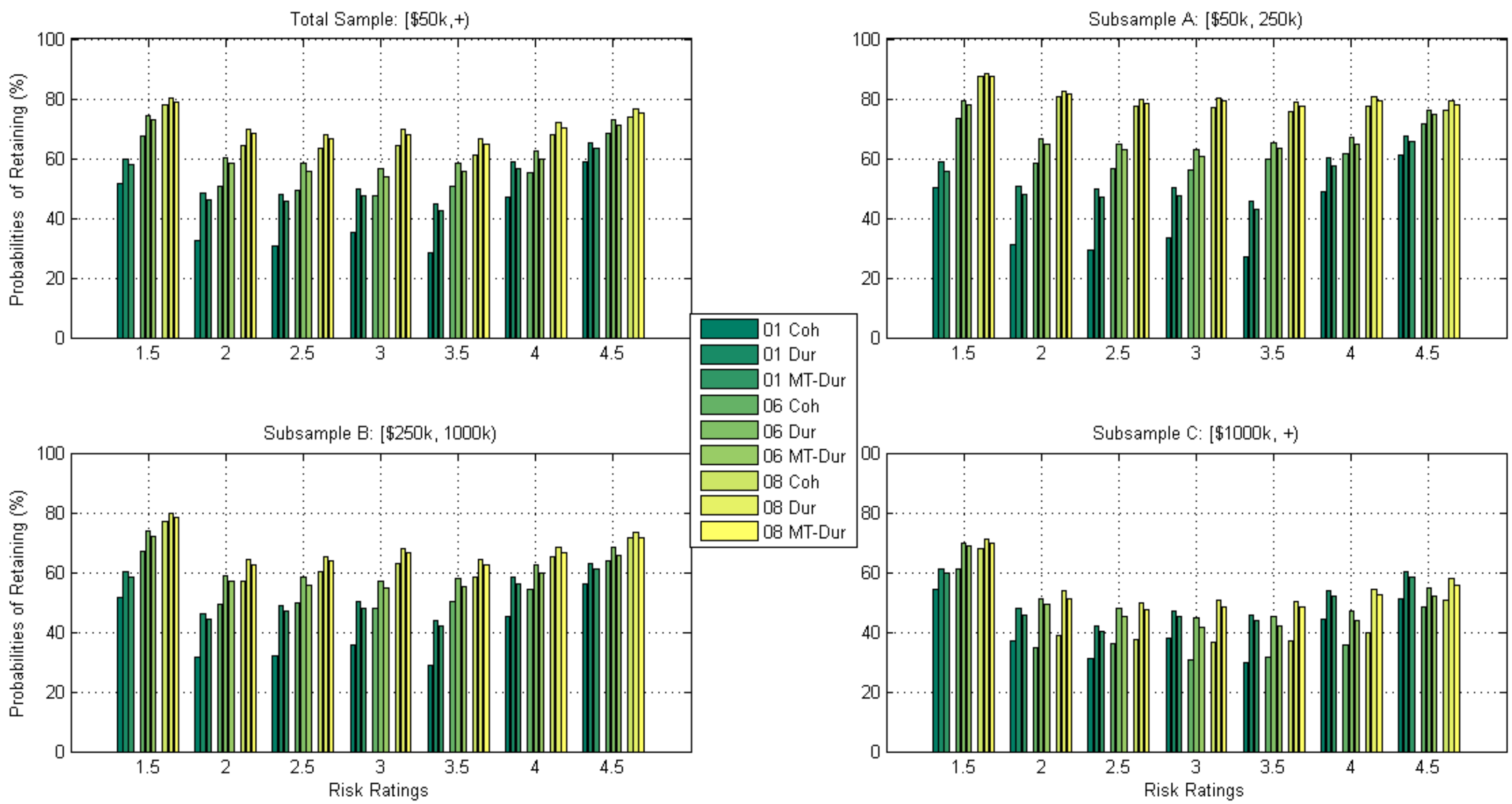
Due to unavailable monitoring data of the *Financing Company*'s internal rating system, the MT-Duration estimates in this thesis do not capture the impact of varying time intervals, which can lead to larger difference between the MT-Duration and the Duration. It is important for banks and regulators to be aware of the impact of the three competing estimation methods for credit risk assessments.

Figure 3.1: Observed Progression Versus Underlying Progression of Credit Rating Status for A Hypothetical Example



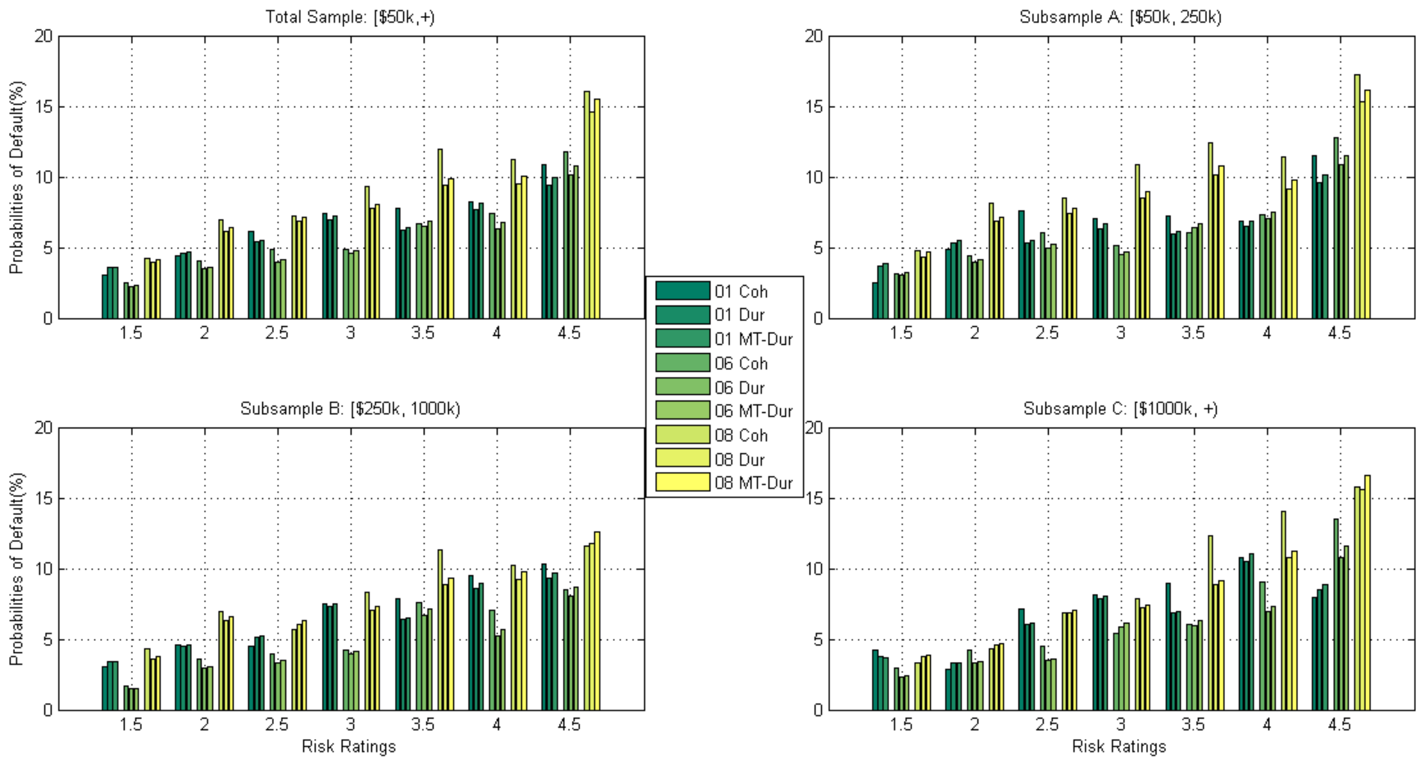
The figure demonstrates how obligors' credit status progressions are viewed under intermittent observations for three obligors described in the example of section 3.3.4. The hypothetical rating system consists of two non-default rating categories (A and B) and a default category D. The dotted vertical lines represent the assessment times, and the solid horizontal arrows represent the true lengths of stay in each rating category.

Figure 3.2: Diagonal Probabilities of Estimated Migration Matrices



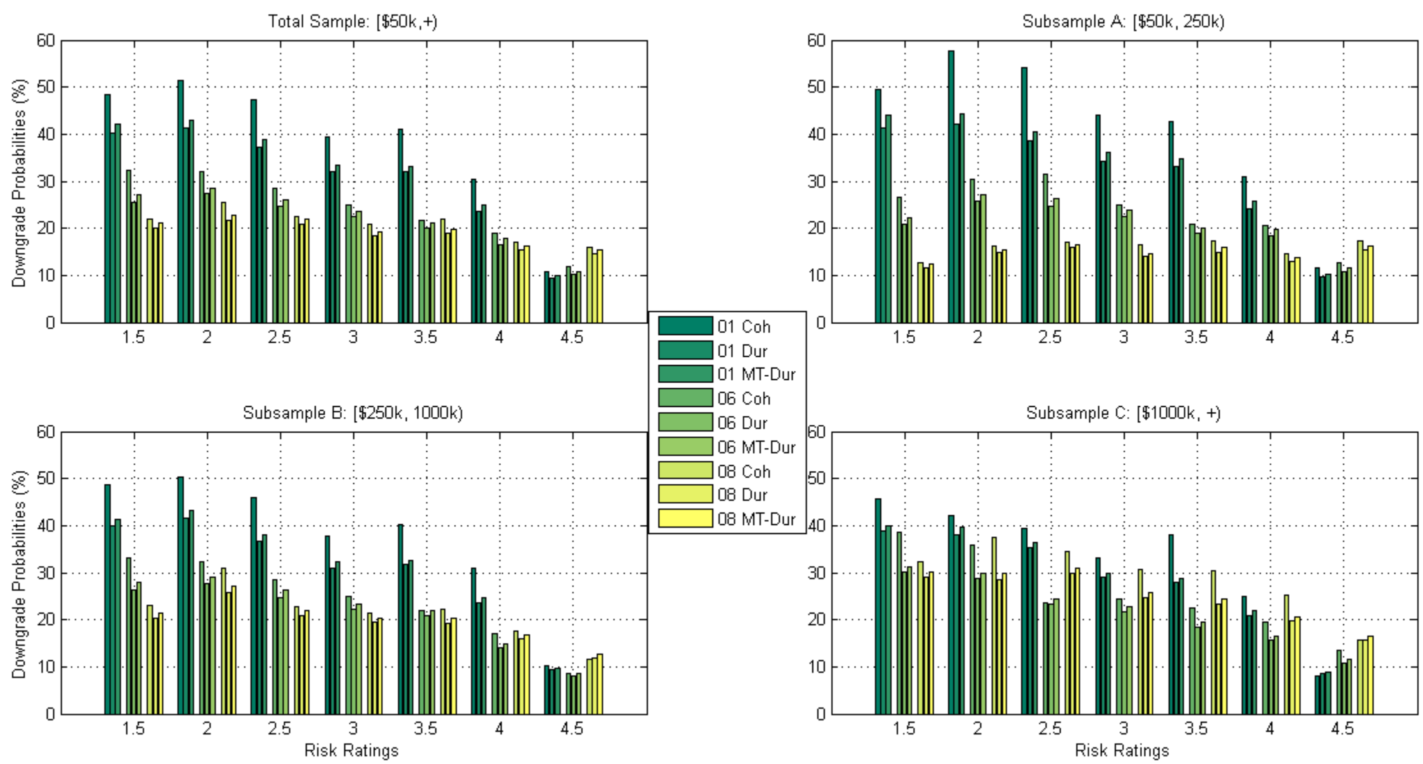
The series of figures compares the diaogal elements of estimated migration matrices by loan size and time. Four sample groups are classified by loan size: subsample A with loan size[$50k, $250k), subsample B with loan size [$250k, $1000k), subsample C with loan size [$1000k, $+\infty$) and the total sample.

Figure 3.3: Default Probabilities of Estimated Migration Matrices



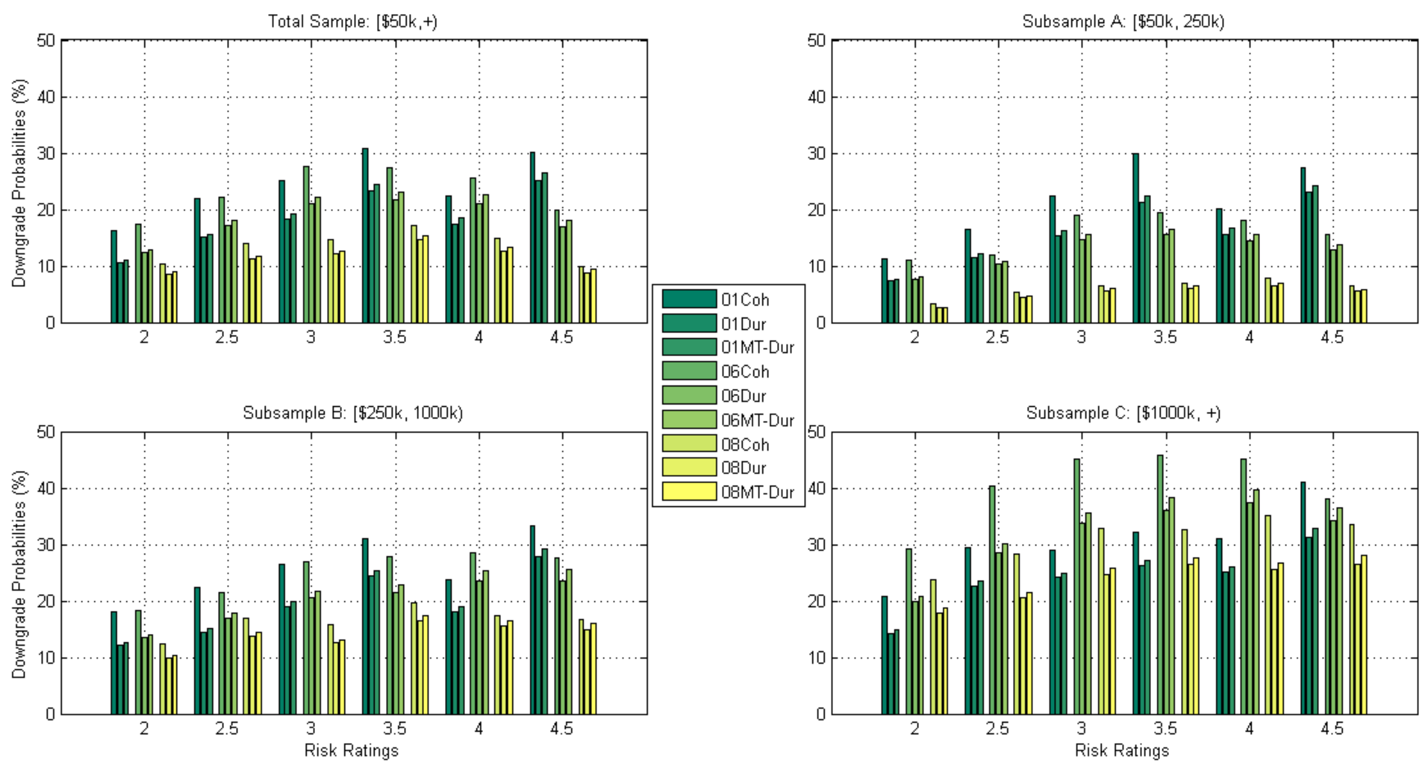
The series of figures compares the Default probabilities of estimated migration matrices by loan size and time. Four sample groups are classified by loan size: subsample A with loan size[$50k, $250k), subsample B with loan size [$250k, $1000k), subsample C with loan size [$1000k, +1) and the total sample.

Figure 3.4: Downgrade Probabilities of Estimated Migration Matrices



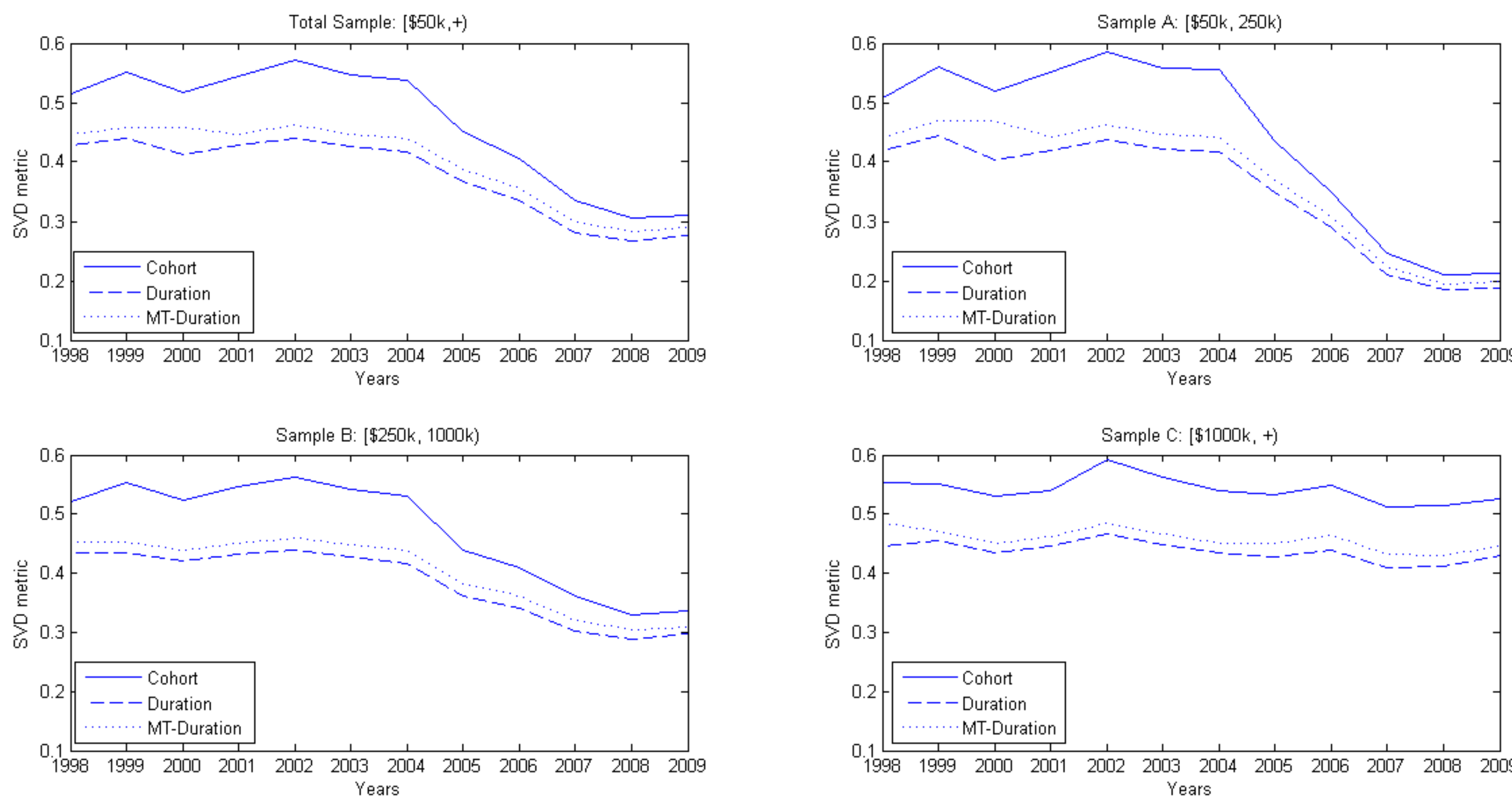
The series of figures compares the downgrade probabilities of estimated migration matrices by loan size and time. The downgrade probabilities for a particular rating category are summed up. Four sample groups are classified by loan size: subsample A with loan size[$50k, $250k), subsample B with loan size [$250k, $1000k), subsample C with loan size [$1000k, +1) and the total sample.

Figure 3.5: Upgrade Probabilities of Estimated Migration Matrices



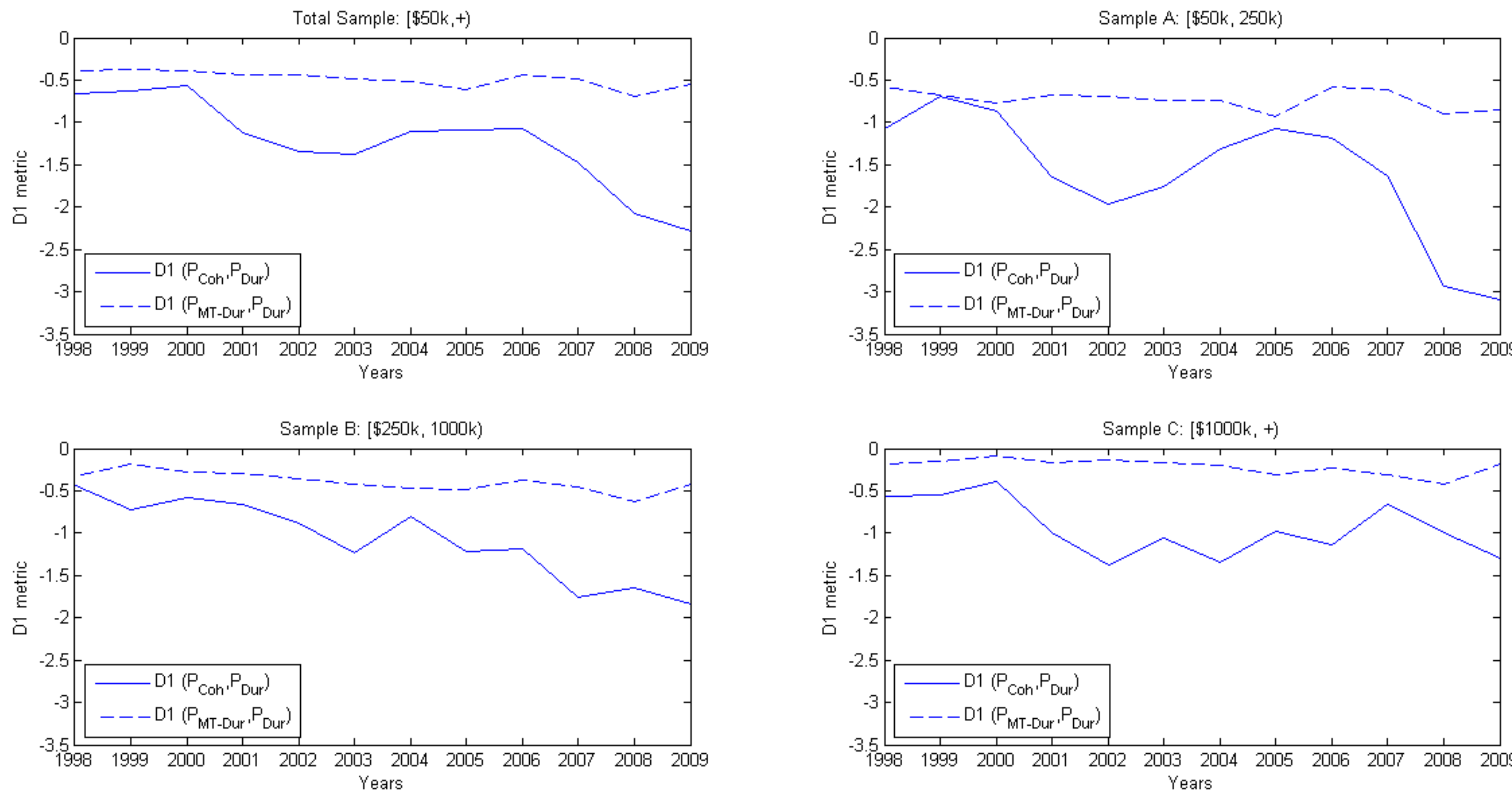
The series of figures compares the upgrade probabilities of estimated migration matrices by loan size and time. The upgrade probabilities for a particular rating category are summed up. Four sample groups are classified by loan size: subsample A with loan size[$50k, $250k), subsample B with loan size [$250k, $1000k), subsample C with loan size [$1000k, $+\infty$) and the total sample.

Figure 3.6: M_{SVD} of Annual Mobility Matrices by Sample Groups, comparing Three Estimation Methods



The series of figures exhibits the annual mobility of the migration matrices using the metric SVD for the period of 1998-2009. Three competing approaches for estimating transition matrices are the Cohort method, the Duration method and the MT-Duration method. Four sample groups are classified by loan size: subsample A with loan size[$50k, $250k), subsample B with loan size [$250k, $1000k), subsample C with loan size [$1000k, $+\infty$) and the total sample.

Figure 3.7: Differences of Annual Migration Matrices among the Three Estimation Methods via Risk-sensitive Metric $D1$ by Sample Groups



The series of figures exhibits the difference among the three competing migration estimation methods via the risk-sensitive metric $D1$ for the period 1998-2009. Three competing approaches for estimating transition matrices are the Cohort method, the Duration method and the MT-Duration method. Four sample groups are classified by loan size: subsample A with loan size[$50k, $250k), subsample B with loan size [$250k, $1000k), subsample C with loan size [$1000k, $+\infty$) and the total sample. The solid line represents $D1(\mathbb{P}_{Coh}, \mathbb{P}_{Dur})$ and slash line represents $D1(\mathbb{P}_{MT-Dur}, \mathbb{P}_{Dur})$.

Table 3.1: Intensity Distribution of Rating Change (%)

Sample Group	Time Interval of Rating Change (Year)			
	[0,0.5)	[0.5,1.5)	[1.5,2.5]	[$\geq$ 2.5)
1998-2004				
sample A	10.7	46.7	17.0	25.6
sample B	12.3	48.5	18.2	21.0
sample C	13.3	48.9	16.1	21.7
2005-2009				
sample A	12.9	38.1	22.1	26.9
sample B	14.9	44.7	19.9	20.4
sample C	15.0	50.7	18.0	16.3

The table describes the distribution of rating change time for the two periods 1998-2004 and 2005-2009 in three subsample groups, including subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$).

Table 3.2: Non-Rated Frequencies in 2006

Rating	Sub-sample A		Sub-sample B		Sub-sample C		Total Sample	
	No.	%	No.	%	No.	%	No.	%
1.5	91	14.4	82	10.4	28	6.5	201	10.9
2	66	9.3	53	8.4	18	6.0	137	8.3
2.5	87	11.4	73	9.3	29	6.7	189	9.6
3	111	11.1	101	10.2	40	7.9	252	10.1
3.5	65	8.9	89	11.7	34	10.8	188	10.4
4	191	12.3	162	13.3	43	10.5	396	12.5
4.5	355	10.6	174	14.0	39	13.8	568	11.7
Total	966	11.1	734	11.5	231	8.6	1931	10.8

The table describes the non-Rated (NR) frequencies in the year 2006 for four sample groups, including subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and the total sample. The data is based on obligor-year observations over 12 years (1998-2009). The portfolio consists of obligors rated from 1.5 (least risk) to 4.5 (riskiest) with 0.5 rating intervals.

Table 3.3: Non-Rated Frequencies over Time

Year	Sub-sample A		Sub-sample B		Sub-sample C		Total Sample	
	No.	%	No.	%	No.	%	No.	%
1998	426	7.5	180	4.7	23	2.4	629	6.0
1999	506	8.6	214	4.9	56	4.8	776	6.8
2000	573	9.7	266	5.6	46	3.4	885	7.4
2001	624	10.3	316	6.2	74	4.7	1014	8.0
2002	662	10.3	400	7.2	74	4.2	1136	8.3
2003	719	10.5	467	8.0	101	5.0	1287	8.7
2004	825	11.2	566	9.2	128	5.7	1519	9.6
2005	869	10.8	632	10.0	188	7.6	1689	10.1
2006	966	11.1	734	11.5	231	8.6	1931	10.8
2007	1090	11.4	705	10.9	247	8.9	2042	10.9
2008	1040	10.4	576	8.9	177	6.0	1793	9.2
2009	1004	10.3	480	7.6	123	4.0	1607	8.4

The table describes the non-Rated (NR) frequencies over 12 years (1998-2009) for four sample groups, including subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and the total sample.

Table 3.4: Time Spent Impact on Diagonal Probabilities for Two Hypothetical Scenarios

Cohort for both scenarios			
Rating	*A*	*B*	*Default*
A	1	0	0
B	0.1000	0.9000	0
Default	0	0	1
Duration for 8 mth stay in B			
Rating	*A*	*B*	*Default*
A	1	0	0
B	0.0983	0.9017	0
Default	0	0	1
Duration for 11 mth stay in B			
Rating	*A*	*B*	*Default*
A	1	0	0
B	0.0959	0.9041	0
Default	0	0	1

The table shows the migration matrices estimated from the Cohort and Duration methods for two hypothetical rating histories: In total 20 obligors in which 10 obligors with rating A and 10 obligors with rating B, we only observe obligor 2 upgrades from rating B to A. One scenario is that obligor 2 upgrade at month 8, and other assumes upgrading occurs at month 11. Rating D represents the default state.

Table 3.5: Migration Matrices: 2001

Panel A: The one-year Cohort transition matrix

Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**51.7**	8.5	6.6	11.3	3.5	10.6	4.7	3.1
2	16.2	**32.4**	8.8	13.5	5.5	11.5	7.9	4.4
2.5	11.1	10.7	**30.9**	14.0	6.4	12.0	8.7	6.2
3	7.5	8.2	9.4	**35.2**	7.4	14.1	10.6	7.4
3.5	4.6	4.4	7.9	13.9	**28.2**	19.2	14.0	7.7
4	1.1	2.1	3.8	9.4	6.2	**47.1**	22.2	8.2
4.5	0.8	1.0	1.9	4.7	6.0	15.8	**58.9**	10.9

Panel B: The one-year Duration transition matrix

Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**59.8**	6.4	5.6	8.2	2.9	8.6	4.9	3.6
2	10.6	**48.3**	7.5	9.5	3.9	8.7	6.9	4.6
2.5	8.0	7.0	**47.8**	10.4	4.9	9.0	7.7	5.4
3	5.8	5.9	6.7	**49.6**	4.9	11.2	9.0	7.0
3.5	3.8	3.3	5.7	10.5	**44.8**	14.4	11.3	6.3
4	1.6	1.9	3.0	6.6	4.3	**58.9**	16.0	7.7
4.5	1.1	1.3	1.8	4.3	4.3	12.4	**65.4**	9.4

Panel C: The one-year MT-Duration transition matrix

Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**57.8**	6.7	6.0	8.7	3.1	9.0	5.2	3.6
2	11.0	**45.9**	7.9	10.0	4.1	9.1	7.3	4.7
2.5	8.4	7.3	**45.5**	10.8	5.1	9.3	8.0	5.5
3	6.0	6.2	7.0	**47.3**	5.1	11.7	9.4	7.2
3.5	4.0	3.4	6.0	11.0	**42.6**	15.0	11.8	6.4
4	1.7	2.0	3.2	7.0	4.6	**56.4**	17.0	8.1
4.5	1.2	1.4	1.9	4.5	4.5	13.0	**63.6**	9.9

The table compares transition matrices estimates using three competing methods for the total sample group with loan size above $50k in 2001. The data is based on obligor months. Figures are in %.

Table 3.6: **Migration Matrices: 2006**

Panel A: The one-year Cohort transition matrix

Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**67.7**	9.0	5.4	4.3	3.2	4.3	3.6	2.5
2	17.3	**50.6**	7.6	6.1	3.9	4.7	5.7	4.1
2.5	13.1	9.1	**49.3**	6.7	4.9	6.6	5.6	4.9
3	10.0	8.4	9.1	**47.6**	5.8	6.5	7.6	4.8
3.5	5.6	6.1	7.0	8.8	**50.9**	7.2	7.8	6.7
4	4.3	4.2	4.2	6.2	6.6	**55.4**	11.6	7.4
4.5	2.0	2.4	2.5	3.5	3.8	5.7	**68.4**	11.8

Panel B: The one-year Duration transition matrix

Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**74.4**	6.7	4.5	3.1	2.3	3.2	3.5	2.3
2	12.4	**60.3**	6.4	4.9	3.3	4.1	5.1	3.5
2.5	9.9	7.2	**58.3**	6.3	4.2	5.2	5.0	4.0
3	7.6	6.4	7.0	**56.5**	5.4	5.8	6.7	4.6
3.5	4.7	5.0	5.4	6.6	**58.2**	6.6	7.0	6.5
4	3.9	3.4	3.6	4.7	5.2	**62.5**	10.2	6.4
4.5	2.1	2.1	2.1	2.8	3.1	4.7	**72.9**	10.1

Panel C: The one-year MT-Duration transition matrix

Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**72.9**	7.2	4.7	3.2	2.5	3.4	3.7	2.3
2	12.9	**58.4**	6.7	5.2	3.4	4.3	5.4	3.6
2.5	10.5	7.6	**55.8**	6.7	4.5	5.5	5.3	4.2
3	8.0	6.8	7.4	**54.0**	5.8	6.1	7.1	4.8
3.5	4.9	5.3	5.7	7.0	**55.8**	6.9	7.4	6.8
4	4.2	3.6	3.9	5.1	5.6	**59.7**	11.0	6.8
4.5	2.2	2.3	2.3	3.0	3.3	5.0	**71.1**	10.8

The table compares transition matrices estimates using three competing methods for the total sample group with loan size above $50k in 2006. The data is based on obligor months. Figures are in %.

Table 3.7: Migration Matrices: 2008

Panel A: The one-year Cohort transition matrix								
Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**78.0**	6.6	3.3	2.2	2.3	1.7	1.6	4.2
2	10.4	**64.2**	5.1	3.9	3.4	3.0	3.0	6.9
2.5	6.8	7.1	**63.6**	5.4	3.6	2.7	3.6	7.2
3	5.1	5.1	4.5	**64.5**	3.8	4.2	3.5	9.3
3.5	3.6	3.8	3.8	6.0	**60.9**	5.4	4.6	11.9
4	2.3	2.9	2.8	3.1	3.7	**68.2**	5.7	11.3
4.5	0.7	1.6	1.5	1.9	1.7	2.7	**74.0**	16.0
Panel B: The one-year Duration transition matrix								
Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**80.1**	5.7	3.1	2.0	1.8	1.5	1.9	4.0
2	8.6	**69.7**	4.2	3.6	2.7	2.3	2.7	6.2
2.5	5.8	5.4	**68.0**	4.6	3.2	2.9	3.3	6.9
3	4.0	4.2	3.8	**69.6**	3.6	3.5	3.5	7.7
3.5	3.2	3.4	3.1	4.9	**66.6**	5.0	4.4	9.4
4	2.1	2.4	2.3	2.7	3.1	**72.1**	5.9	9.5
4.5	0.8	1.3	1.3	1.6	1.4	2.4	**76.5**	14.6
Panel C: The one-year MT-Duration transition matrix								
Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**78.9**	6.1	3.3	2.2	1.9	1.6	2.0	4.1
2	9.0	**68.2**	4.5	3.8	2.8	2.4	2.9	6.4
2.5	6.1	5.7	**66.4**	4.9	3.4	3.0	3.4	7.1
3	4.2	4.5	4.0	**68.1**	3.8	3.7	3.7	8.0
3.5	3.3	3.6	3.2	5.2	**64.7**	5.3	4.7	9.9
4	2.2	2.6	2.4	2.9	3.3	**70.3**	6.3	10.0
4.5	0.8	1.4	1.4	1.7	1.5	2.6	**75.2**	15.5

The table compares transition matrices estimates using three competing methods for the total sample group with loan size above $50k in 2008. The data is based on obligor months. Figures are in %.

Table 3.8: **Estimated Diagonal Probabilities for Different Estimation Methods**

	2001			2006			2008		
Rating	Coh	Dur	MT-Dur	Coh	Dur	MT-Dur	Coh	Dur	MT-Dur
Total Sample									
1.5	51.7	59.8	57.8	67.7	74.4	72.9	78.0	80.1	78.9
2	32.4	48.3	45.9	50.6	60.3	58.4	64.2	69.7	68.2
2.5	30.9	47.8	45.5	49.3	58.3	55.8	63.6	68.0	66.4
3	35.2	49.6	47.3	47.6	56.5	54.0	64.5	69.6	68.1
3.5	28.2	44.8	42.6	50.9	58.2	55.8	60.9	66.6	64.7
4	47.1	58.9	56.4	55.4	62.5	59.7	68.2	72.1	70.3
4.5	58.9	65.4	63.6	68.4	72.9	71.1	74.0	76.5	75.2
Sample A									
1.5	50.4	58.7	55.8	73.5	79.2	77.7	87.3	88.5	87.6
2	31.1	50.7	48.1	58.6	66.5	64.8	80.6	82.6	81.8
2.5	29.4	49.8	47.3	56.4	65.0	62.8	77.6	79.6	78.6
3	33.6	50.3	47.5	56.1	62.9	60.7	76.9	80.4	79.3
3.5	27.2	45.5	42.9	59.7	65.3	63.5	75.7	78.9	77.6
4	48.9	60.3	57.4	61.6	67.1	64.7	77.4	80.7	79.2
4.5	61.1	67.4	65.6	71.6	76.2	74.8	76.3	79.1	78.0
Sample B									
1.5	51.4	60.1	58.6	66.9	73.8	72.2	76.9	79.7	78.5
2	31.5	46.3	44.1	49.5	58.8	57.0	56.9	64.5	62.6
2.5	31.8	48.9	46.8	50.0	58.4	55.9	60.4	65.2	63.6
3	35.8	50.1	48.0	48.0	57.2	54.9	62.9	68.0	66.6
3.5	28.7	43.9	41.9	50.2	57.9	55.2	58.3	64.2	62.3
4	45.3	58.3	56.2	54.3	62.5	59.6	65.1	68.5	66.7
4.5	56.3	62.9	61.1	63.9	68.4	65.8	71.7	73.3	71.4
Sample C									
1.5	54.2	61.1	59.9	61.3	69.8	68.7	67.7	71.0	69.8
2	37.0	48.0	45.6	34.9	51.3	49.3	38.7	53.7	51.4
2.5	31.3	42.2	40.1	36.0	48.1	45.4	37.3	49.7	47.5
3	37.8	46.9	45.2	30.6	44.6	41.7	36.5	50.5	48.2
3.5	29.8	45.8	43.9	31.7	45.4	42.2	37.1	50.2	48.2
4	44.1	54.0	52.0	35.5	46.9	43.9	39.6	54.5	52.5
4.5	51.0	60.2	58.4	48.4	55.0	51.8	50.6	57.8	55.5

The table exhibits the estimated diagonal probabilities for the three competing estimates over the years 2001, 2006 and 2008. Four sample groups are classified by loan size: subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and the total sample.

Table 3.9: **Estimated Probability of Default (PDs) for Different Estimation Methods**

	2001			2006			2008		
Rating	Coh	Dur	MT-Dur	Coh	Dur	MT-Dur	Coh	Dur	MT-Dur
Total Sample									
1.5	3.1	3.6	3.6	2.5	2.3	2.3	4.2	4.0	4.1
2	4.4	4.6	4.7	4.1	3.5	3.6	6.9	6.2	6.4
2.5	6.2	5.4	5.5	4.9	4.0	4.2	7.2	6.9	7.1
3	7.4	7.0	7.2	4.8	4.6	4.8	9.3	7.7	8.0
3.5	7.7	6.3	6.4	6.7	6.5	6.8	11.9	9.4	9.9
4	8.2	7.7	8.1	7.4	6.4	6.8	11.3	9.5	10.0
4.5	10.9	9.4	9.9	11.8	10.1	10.8	16.0	14.6	15.5
Sample A									
1.5	2.5	3.7	3.8	3.2	3.0	3.2	4.8	4.4	4.7
2	4.9	5.3	5.5	4.4	4.0	4.1	8.1	6.8	7.1
2.5	7.6	5.3	5.5	6.1	4.9	5.2	8.5	7.4	7.7
3	7.0	6.3	6.7	5.2	4.5	4.6	10.9	8.5	8.9
3.5	7.2	6.0	6.2	6.0	6.4	6.7	12.4	10.2	10.8
4	6.9	6.5	6.9	7.3	7.0	7.5	11.4	9.1	9.8
4.5	11.5	9.5	10.1	12.8	10.8	11.5	17.3	15.4	16.2
Sample B									
1.5	3.1	3.4	3.4	1.7	1.5	1.5	4.4	3.6	3.8
2	4.6	4.5	4.6	3.6	2.9	3.0	6.9	6.3	6.6
2.5	4.5	5.2	5.2	4.0	3.3	3.5	5.7	6.1	6.3
3	7.5	7.3	7.5	4.3	4.0	4.1	8.3	7.1	7.3
3.5	7.9	6.4	6.5	7.6	6.7	7.1	11.3	8.9	9.3
4	9.5	8.6	9.0	7.0	5.3	5.6	10.2	9.3	9.7
4.5	10.3	9.3	9.7	8.5	8.0	8.7	11.6	11.8	12.6
Sample C									
1.5	4.2	3.7	3.7	3.0	2.4	2.4	3.3	3.8	3.8
2	2.9	3.3	3.3	4.2	3.3	3.4	4.3	4.5	4.6
2.5	7.1	6.1	6.1	4.5	3.5	3.6	6.9	6.9	7.1
3	8.1	7.9	8.1	5.4	5.9	6.2	7.9	7.2	7.4
3.5	8.9	6.9	7.0	6.0	6.0	6.3	12.3	8.9	9.2
4	10.8	10.5	11.0	9.0	6.9	7.3	14.0	10.8	11.2
4.5	7.9	8.5	8.8	13.5	10.8	11.6	15.8	15.6	16.6

The table exhibits the estimated probabilities of default using the three competing methods over the years 2001, 2006 and 2008. Four sample groups are classified by loan size: subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and the total sample.

Table 3.10: Estimated Downgrade Probabilities for Different Estimation Methods

	2001			2006			2008		
Rating	Coh	Dur	MT-Dur	Coh	Dur	MT-Dur	Coh	Dur	MT-Dur
Total Sample									
1.5	48.3	40.2	42.2	32.3	25.6	27.1	22.0	19.9	21.1
2	51.4	41.2	43.0	32.1	27.4	28.6	25.4	21.7	22.8
2.5	47.3	37.2	38.8	28.6	24.7	26.1	22.5	20.8	21.8
3	39.5	32.1	33.5	24.8	22.5	23.7	20.9	18.4	19.2
3.5	41.0	31.9	33.1	21.6	20.0	21.2	21.9	18.8	19.8
4	30.4	23.6	25.1	19.0	16.6	17.8	17.0	15.3	16.3
4.5	10.9	9.4	9.9	11.8	10.1	10.8	16.0	14.6	15.5
Sample A									
1.5	49.6	41.3	44.2	26.5	20.8	22.3	12.7	11.5	12.4
2	57.6	42.1	44.2	30.3	25.9	27.2	16.2	14.8	15.5
2.5	54.1	38.7	40.6	31.6	24.8	26.3	17.1	15.8	16.6
3	44.1	34.3	36.2	24.9	22.5	23.8	16.6	14.0	14.7
3.5	42.8	33.2	34.8	20.9	19.1	20.0	17.4	15.0	15.9
4	30.9	24.1	25.9	20.5	18.5	19.8	14.6	12.9	13.9
4.5	11.5	9.5	10.1	12.8	10.8	11.5	17.3	15.4	16.2
Sample B									
1.5	48.6	39.9	41.4	33.1	26.2	27.8	23.1	20.3	21.5
2	50.4	41.6	43.2	32.2	27.7	29.0	30.8	25.6	27.0
2.5	45.8	36.6	38.0	28.5	24.7	26.2	22.7	21.0	21.9
3	37.7	30.9	32.2	25.1	22.1	23.3	21.3	19.4	20.2
3.5	40.3	31.7	32.7	22.0	20.7	22.0	22.2	19.3	20.3
4	31.0	23.6	24.7	17.1	13.9	15.0	17.5	16.0	16.9
4.5	10.3	9.3	9.7	8.5	8.0	8.7	11.6	11.8	12.6
Sample C									
1.5	45.8	38.9	40.1	38.7	30.2	31.3	32.3	29.0	30.2
2	42.2	37.9	39.6	35.9	28.8	29.9	37.5	28.4	29.9
2.5	39.3	35.2	36.4	23.6	23.3	24.5	34.4	29.8	31.1
3	33.2	28.9	29.8	24.4	21.6	22.7	30.7	24.8	25.9
3.5	38.1	28.0	28.8	22.4	18.5	19.6	30.3	23.4	24.3
4	24.9	20.9	21.8	19.4	15.7	16.6	25.3	19.8	20.7
4.5	7.9	8.5	8.8	13.5	10.8	11.6	15.8	15.6	16.6

The table exhibits the estimated probabilities of downgrade using the three competing methods over the years 2001, 2006 and 2008. Downgrade probability of each rating in the table is the sum of transition probabilities of this rating to all other downgrading ratings. Four sample groups are classified by loan size: subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and the total sample.

Table 3.11: Estimated Upgrade Probabilities for Different Estimation Methods

	2001			2006			2008		
Rating	Coh	Dur	MT-Dur	Coh	Dur	MT-Dur	Coh	Dur	MT-Dur
Total Sample									
1.5	NA	NA	NA	NA	NA	NA	NA	NA	NA
2	16.2	10.6	11.0	17.3	12.4	12.9	10.4	8.6	9.0
2.5	21.9	15.0	15.7	22.2	17.1	18.1	13.9	11.2	11.8
3	25.2	18.3	19.2	27.5	21.0	22.3	14.7	12.1	12.7
3.5	30.8	23.3	24.3	27.5	21.7	23.0	17.2	14.6	15.4
4	22.5	17.5	18.6	25.6	20.9	22.5	14.8	12.5	13.4
4.5	30.2	25.2	26.5	19.8	17.0	18.1	10.0	8.8	9.4
Sample A									
1.5	NA	NA	NA	NA	NA	NA	NA	NA	NA
2	11.3	7.3	7.6	11.1	7.6	8.0	3.2	2.6	2.7
2.5	16.5	11.6	12.1	12.0	10.2	10.9	5.3	4.5	4.7
3	22.3	15.4	16.3	19.0	14.6	15.5	6.5	5.7	6.0
3.5	30.0	21.3	22.3	19.4	15.7	16.5	6.8	6.1	6.5
4	20.2	15.6	16.7	18.0	14.4	15.5	7.9	6.4	6.9
4.5	27.4	23.0	24.3	15.6	12.9	13.7	6.4	5.5	5.8
Sample B									
1.5	NA	NA	NA	NA	NA	NA	NA	NA	NA
2	18.2	12.1	12.6	18.3	13.4	14.0	12.3	9.9	10.4
2.5	22.3	14.5	15.2	21.5	16.9	17.9	17.0	13.8	14.4
3	26.5	19.0	19.8	26.9	20.6	21.8	15.8	12.6	13.1
3.5	31.0	24.5	25.4	27.8	21.4	22.7	19.6	16.5	17.4
4	23.7	18.1	19.1	28.6	23.5	25.4	17.4	15.5	16.4
4.5	33.4	27.9	29.2	27.5	23.6	25.5	16.7	14.9	16.0
Sample C									
1.5	NA	NA	NA	NA	NA	NA	NA	NA	NA
2	20.8	14.1	14.8	29.2	20.0	20.8	23.8	17.8	18.8
2.5	29.5	22.6	23.5	40.4	22.6	30.2	28.3	20.5	21.4
3	29.0	24.1	25.0	45.0	24.1	35.6	32.8	24.7	25.9
3.5	32.1	26.2	27.2	45.9	26.2	38.3	32.6	26.4	27.5
4	31.0	25.1	26.1	45.1	25.1	39.6	35.1	25.7	26.8
4.5	41.1	31.3	32.8	38.1	31.3	36.6	33.6	26.6	28.0

The table exhibits the estimated probabilities of upgrade using the three competing methods over the years 2001, 2006 and 2008. Upgrade probability of each rating in the table is the sum of transition probabilities of this rating to all other upgrade ratings. Four sample groups are classified by loan size: subsample A with loan size [\$50k, \$250k), subsample B with loan size [\$250k, \$1000k), subsample C with loan size [\$1000k, $+\infty$) and the total sample.

Table 3.12: **Ratings Distribution of Sample Portfolios**

Rating	Total Sample	Sample A	Sample B	Sample C
2001				
1.5	1370	512	588	270
2	981	361	435	185
2.5	1103	398	472	233
3	2029	830	876	323
3.5	1309	542	595	172
4	2810	1511	1068	231
4.5	3110	1898	1054	158
Total	12712	6052	5088	1572
2006				
1.5	1846	630	785	431
2	1641	706	633	302
2.5	1975	762	781	432
3	2502	1004	994	504
3.5	1807	730	762	315
4	3173	1549	1215	409
4.5	4872	3350	1239	283
Total	17816	8731	6409	2676
2008				
1.5	2994	1052	1180	762
2	2259	1054	756	449
2.5	2097	959	734	404
3	2395	1059	933	403
3.5	1991	800	827	364
4	2644	1352	988	304
4.5	5022	3729	1021	272
Total	19402	10005	6439	2958

The table demonstrates ratings distribution of sample portfolios. Sample A is the obligors with loans size [\$50k,\$250k), Sample B is the obligors with loans size [\$250k,\$1000k) and Sample C is the obligors with loans size [\$1000k,$+\infty$).

Table 3.13: **Credit VaR: Comparing Estimation Methods (2001)**

	Cohort	Duration	MT-Duration	$\%\frac{Cohort}{Duration}$	$\%\frac{MT-Duration}{Duration}$
Total Sample					
Mean	161.91	148.81	154.78	108.80	104.01
Std.Dev	65.20	62.24	64.74	104.74	104.01
VaR(99%)	355.62	327.57	344.62	108.56	105.20
VaR(99.9%)	451.46	432.55	455.48	104.37	105.30
Sample A					
Mean	24.22	21.33	22.81	113.56	106.92
Std.Dev	9.62	8.78	9.28	109.60	105.66
VaR(99%)	52.71	47.07	50.47	111.98	107.23
VaR(99.9%)	68.49	60.36	64.72	113.47	107.21
Sample B					
Mean	71.54	66.88	69.04	106.97	103.24
Std.Dev	28.68	27.36	28.06	104.83	102.56
VaR(99%)	159.72	149.10	150.87	107.13	101.19
VaR(99.9%)	201.89	179.85	194.80	112.26	108.31
Sample C					
Mean	72.54	67.27	68.62	107.83	102.00
Std.Dev	31.07	29.36	28.95	105.82	98.61
VaR(99%)	164.20	157.21	152.43	104.45	96.96
VaR(99.9%)	218.20	203.50	198.63	107.22	97.60

Credit default loss as computed by one-factor version of CreditMetrics. The sample portfolios are as described in Table 3.12. Monte Carlo simulations (100,000 trials) are performed to obtain the portfolio default loss distribution for a one-year horizon. The first three columns are in $M.

Table 3.14: **Credit VaR: Comparing Estimation Methods (2006)**

	Cohort	Duration	MT-Duration	$\%\frac{Cohort}{Duration}$	$\%\frac{MT-Duration}{Duration}$
Total Sample					
Mean	225.68	199.44	208.67	113.15	104.63
Std.Dev	94.79	86.25	89.09	109.90	103.29
VaR(99%)	511.41	461.65	472.25	110.78	102.30
VaR(99.9%)	628.68	607.25	633.92	103.53	104.39
Sample A					
Mean	30.28	26.81	28.21	112.93	105.24
Std.Dev	11.73	10.71	11.14	109.43	103.93
VaR(99%)	64.29	57.38	60.72	112.06	105.83
VaR(99.9%)	81.20	74.41	78.24	109.12	105.15
Sample B					
Mean	63.43	54.70	58.19	115.96	106.38
Std.Dev	27.46	24.33	25.78	112.84	105.96
VaR(99%)	145.61	130.28	137.18	111.77	105.30
VaR(99.9%)	189.60	166.96	182.29	113.56	109.18
Sample C					
Mean	133.15	115.19	119.44	115.59	103.69
Std.Dev	56.66	50.79	52.97	111.56	104.29
VaR(99%)	301.51	268.98	282.74	112.09	105.12
VaR(99.9%)	374.82	335.80	354.34	111.62	105.52

Credit default loss as computed by one-factor version of CreditMetrics. The sample portfolios are as described in Table 3.12. Monte Carlo simulations (100,000 trials) are performed to obtain the portfolio default loss distribution for a one-year horizon. The first three columns are in $M.

Table 3.15: **Credit VaR: Comparing Estimation Methods (2008)**

	Cohort	Duration	MT-Duration	$\% \frac{Cohort}{Duration}$	$\% \frac{MT-Duration}{Duration}$
Total Sample					
Mean	390.63	355.69	357.69	109.82	100.56
Std.Dev	149.31	134.38	139.94	111.11	104.14
VaR(99%)	824.50	741.33	791.02	111.22	106.70
VaR(99.9%)	1028.55	942.98	985.78	109.07	104.54
Sample A					
Mean	53.47	47.65	48.01	112.22	100.76
Std.Dev	18.69	16.57	17.22	112.74	103.89
VaR(99%)	107.81	94.05	97.64	114.62	103.81
VaR(99.9%)	123.84	116.93	116.69	105.91	99.79
Sample B					
Mean	97.12	91.74	92.37	105.87	100.68
Std.Dev	38.08	35.69	36.56	106.68	102.44
VaR(99%)	209.28	193.06	198.63	108.40	102.88
VaR(99.9%)	266.00	248.03	255.25	107.24	102.91
Sample C					
Mean	213.51	201.42	201.97	106.00	100.28
Std.Dev	83.85	79.75	82.08	105.13	102.92
VaR(99%)	462.32	436.02	443.34	106.03	101.68
VaR(99.9%)	576.05	540.49	555.03	106.58	102.69

Credit default loss as computed by one-factor version of CreditMetrics. The sample portfolios are as described in Table 3.12. Monte Carlo simulations (100,000 trials) are performed to obtain the portfolio default loss distribution for a one-year horizon. The first three columns are in $M.

Table 3.16: Path Dependent Migration Matrices (2001)

Up-Momentum-Matrix									
Raing	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	**52.63**	11.58	7.37	11.58	3.16	5.26	5.26	3.16	95
2	18.06	**37.50**	11.11	15.28	2.78	11.11	1.39	2.78	72
2.5	14.86	13.51	**31.08**	13.51	6.76	6.76	6.76	6.76	74
3	6.25	7.50	11.25	**41.25**	6.25	16.25	6.25	5.00	80
3.5	2.94	2.94	0.00	26.47	**26.47**	23.53	8.82	8.82	34
4	4.17	0.00	4.17	8.33	12.50	**41.67**	12.50	16.67	24
4.5	-	-	-	-	-	-	-	-	0
							Avg PD	5.54	
Maintain-Momentum-Matrix									
Rating	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	**51.33**	8.85	8.85	15.93	0.88	7.08	1.77	5.31	113
2	21.74	**32.61**	8.70	13.04	6.52	10.87	4.35	2.17	46
2.5	20.69	17.24	**25.86**	17.24	6.90	6.90	0.00	5.17	58
3	9.65	8.77	9.65	**31.58**	6.14	15.79	7.02	11.40	114
3.5	1.72	10.34	3.45	6.90	**25.86**	25.86	15.52	10.34	58
4	0.00	1.32	2.63	9.21	11.84	**50.00**	9.21	15.79	76
4.5	1.56	0.00	4.69	4.69	3.13	21.88	**53.13**	10.94	64
							Avg PD	9.07	
Down-Momentum-Matrix									
Rating	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	**-**	-	-	-	-	-	-	-	0
2	41.18	**35.29**	5.88	11.76	5.88	0.00	0.00	0.00	17
2.5	26.19	11.90	**28.57**	11.90	0.00	9.52	4.76	7.14	42
3	10.20	20.41	10.20	**40.82**	6.12	6.12	2.04	4.08	49
3.5	6.82	6.82	6.82	22.73	**29.55**	15.91	2.27	9.09	44
4	3.37	1.12	5.62	15.73	11.24	**32.58**	19.10	11.24	89
4.5	1.35	1.35	1.35	12.16	10.81	10.81	**54.05**	8.11	74
							Avg PD	7.94	
Unconditional Matrix									
Rating	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	**51.92**	10.10	8.17	13.94	1.92	6.25	3.37	4.33	208
2	22.22	**35.56**	9.63	14.07	4.44	9.63	2.22	2.22	135
2.5	19.54	14.37	**28.74**	14.37	5.17	7.47	4.02	6.32	174
3	8.64	10.70	10.29	**36.63**	6.17	13.99	5.76	7.82	243
3.5	3.68	7.35	3.68	16.91	**27.21**	22.06	9.56	9.56	136
4	2.12	1.06	4.23	12.17	11.64	**40.74**	14.29	13.76	189
4.5	1.45	0.72	2.90	8.70	7.25	15.94	53.62	9.42	138
							Avg PD	7.69	

The table exhibits the migration matrices for the year 2001 based on obligors' rating experience in previous year: upgrade, downgrade or no change. Note that the Unconditional Matrix means that the migration matrix is not conditional on previous rating change experience. Figures in the table are in %.

Table 3.17: Path Dependent Migration Matrices (2006)

Up-Momentum-Matrix									
Raing	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	**61.15**	17.83	8.28	3.18	3.82	1.91	0.64	3.18	157
2	31.58	**28.42**	15.79	7.37	7.37	2.11	2.11	5.26	95
2.5	25.24	15.53	**33.01**	5.83	6.80	5.83	0.97	6.80	103
3	15.38	17.09	13.68	**26.50**	6.84	9.40	2.56	8.55	117
3.5	8.33	4.17	18.75	12.50	**31.25**	10.42	4.17	10.42	48
4	3.70	3.70	7.41	3.70	3.70	**51.85**	22.22	3.70	27
4.5	-	-	-	-	-	-	-	-	0
							Avg PD	6.03	
Maintain-Momentum-Matrix									
Rating	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	**64.41**	9.04	5.65	6.78	4.52	5.08	2.82	1.69	177
2	25.53	**44.68**	12.77	7.45	2.13	2.13	4.26	1.06	94
2.5	25.19	16.79	**35.11**	9.16	3.82	5.34	2.29	2.29	131
3	16.22	9.46	16.89	**31.76**	6.76	8.11	6.76	4.05	148
3.5	5.68	7.95	12.50	21.59	**26.14**	11.36	7.95	6.82	88
4	6.40	11.20	4.00	11.20	12.00	**34.40**	10.40	10.40	125
4.5	5.32	3.19	6.38	8.51	5.32	11.70	**45.74**	13.83	94
							Avg PD	5.25	
Down-Momentum-Matrix									
Rating	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	-	-	-	-	-	-	-	-	0
2	37.84	**29.73**	10.81	8.11	2.70	5.41	2.70	2.70	37
2.5	23.91	14.13	**38.04**	6.52	8.70	3.26	1.09	4.35	92
3	15.89	12.15	17.76	**35.51**	7.48	3.74	1.87	5.61	107
3.5	7.50	12.50	13.75	16.25	**35.00**	5.00	3.75	6.25	80
4	13.57	6.43	6.43	7.14	14.29	**35.71**	9.29	7.14	140
4.5	6.73	6.73	0.96	8.65	6.73	10.58	**48.08**	11.54	104
							Avg PD	6.79	
Unconditional Matrix									
Rating	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	**62.87**	13.17	6.89	5.09	4.19	3.59	1.80	2.40	334
2	30.09	**35.40**	13.72	7.52	4.42	2.65	3.10	3.10	226
2.5	24.85	15.64	**35.28**	7.36	6.13	4.91	1.53	4.29	326
3	15.86	12.63	16.13	**31.18**	6.99	7.26	4.03	5.91	372
3.5	6.94	8.80	14.35	17.59	**30.56**	8.80	5.56	7.41	216
4	9.59	8.22	5.48	8.56	12.33	**36.64**	10.96	8.22	292
4.5	6.06	5.05	3.54	8.59	6.06	11.11	46.97	12.63	198
							Avg PD	5.91	

The table exhibits the migration matrices for the year 2006 based on obligors' rating experience in previous year: upgrade, downgrade or no change. Note that the Unconditional Matrix means that the migration matrix is not conditional on previous rating change experience. Figures in the table are in %.

Table 3.18: Path Dependent Migration Matrices (2008)

Up-Momentum-Matrix									
Raing	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	**65.68**	11.81	5.17	4.80	7.38	0.74	1.48	2.95	271
2	23.70	**37.78**	12.59	5.93	6.67	7.41	2.22	3.70	135
2.5	9.28	11.34	**48.45**	10.31	7.22	4.12	2.06	7.22	97
3	13.51	12.16	9.46	**32.43**	12.16	6.76	9.46	4.05	74
3.5	6.52	4.35	4.35	8.70	**28.26**	23.91	8.70	15.22	46
4	0.00	0.00	0.00	13.04	8.70	**52.17**	13.04	13.04	23
4.5	-	-	-	-	-	-	-	-	0
							Avg PD	5.11	
Maintain-Momentum-Matrix									
Rating	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	**70.68**	10.41	3.56	4.11	3.56	1.64	2.19	3.84	365
2	22.30	**39.19**	11.49	8.78	4.05	3.38	6.08	4.73	148
2.5	20.83	14.17	**30.00**	8.33	10.83	5.83	3.33	6.67	120
3	13.64	12.12	7.58	**35.61**	6.06	9.09	3.03	12.88	132
3.5	6.73	4.81	7.69	10.58	**41.35**	11.54	8.65	8.65	104
4	7.37	9.47	10.53	5.26	8.42	**35.79**	12.63	10.53	95
4.5	1.77	2.65	3.54	5.31	4.42	15.04	**53.98**	13.27	113
							Avg PD	7.43	
Down-Momentum-Matrix									
Rating	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	-	-	-	-	-	-	-	-	0
2	29.85	**34.33**	13.43	2.99	2.99	8.96	4.48	2.99	67
2.5	14.46	18.07	**27.71**	15.66	10.84	7.23	1.20	4.82	83
3	8.24	11.76	10.59	**41.18**	8.24	8.24	3.53	8.24	85
3.5	6.36	10.00	7.27	14.55	**37.27**	6.36	3.64	14.55	110
4	1.06	8.51	9.57	6.38	12.77	**42.55**	5.32	13.83	94
4.5	5.68	6.82	3.41	3.41	10.23	10.23	**44.32**	15.91	88
							Avg PD	10.63	
Unconditional Matrix									
Rating	1.5	2	2.5	3	3.5	4	4.5	D	Obligors Yrs
1.5	**68.55**	11.01	4.25	4.40	5.19	1.26	1.89	3.46	636
2	24.29	**37.71**	12.29	6.57	4.86	6.00	4.29	4.00	350
2.5	15.33	14.33	**35.33**	11.00	9.67	5.67	2.33	6.33	300
3	12.03	12.03	8.93	**36.43**	8.25	8.25	4.81	9.28	291
3.5	6.54	6.92	6.92	11.92	**37.31**	11.54	6.54	12.31	260
4	3.77	8.02	8.96	6.60	10.38	**40.57**	9.43	12.26	212
4.5	3.48	4.48	3.48	4.48	6.97	12.94	49.75	14.43	201
							Avg PD	7.51	

The table exhibits the migration matrices for the year 2008 based on obligors' rating experience in previous year: upgrade, downgrade or no change. Note that the Unconditional Matrix means that the migration matrix is not conditional on previous rating change experience. Figures in the table are in %.

Table 3.19: Up- and Down-grade Probabilities for Momentum Matrices

Ratings	Down-Momentum		Maintain-Momentum		Up-Momentum	
	Upgrade	Downgrade	Upgrade	Downgrade	Upgrade	Downgrade
2001						
1.5	-	-	-	48.7	-	47.4
2	41.2	23.5	21.7	45.7	18.1	44.4
2.5	38.1	33.3	37.9	36.2	28.4	40.5
3	40.8	18.4	28.1	40.4	25.0	33.8
3.5	43.2	27.3	22.4	51.7	32.4	41.2
4	37.1	30.3	25.0	25.0	29.2	29.2
4.5	37.8	8.1	35.9	10.9	-	-
2006						
1.5	-	-	-	35.6	-	38.9
2	37.8	32.4	25.5	29.8	31.6	40.0
2.5	38.0	23.9	42.0	22.9	40.8	26.2
3	45.8	18.7	42.6	25.7	46.2	27.4
3.5	50.0	15.0	47.7	26.1	43.8	25.0
4	47.9	16.4	44.8	20.8	22.2	25.9
4.5	40.4	11.5	40.4	13.8	-	-
2008						
1.5	-	-	-	29.3	-	34.3
2	29.9	35.8	22.3	38.5	23.7	38.5
2.5	32.5	39.8	35.0	35.0	20.6	30.9
3	30.6	28.2	33.3	31.1	35.1	32.4
3.5	38.2	24.5	29.8	28.8	23.9	47.8
4	38.3	19.1	41.1	23.2	21.7	26.1
4.5	39.8	15.9	32.7	13.3	-	-

The table presents the sum of upgrade probabilities and downgrade probabilities for each rating of momentum-based migration matrix for the years 2001, 2006 and 2008. The path dependent momentum matrices refer to Table 3.16-3.18. Figures in the table are in %.

Table 3.20: Up and Down Probabilities Differences between Momentum Matrices and Unconditional Matrix

Rating	Down-Momentum		Maintain-Momentum		Up-Momentum	
	up% Δ	down% Δ	up% Δ	down%Δ	up%Δ	down% Δ
2001						
1.5	-	-	-	0.6	-	-0.7
2	19.0	-18.7	-0.5	3.4	-4.2	2.2
2.5	4.2	-4.0	4.0	-1.1	-5.5	3.2
3	11.2	-15.4	-1.6	6.6	-4.6	0.0
3.5	11.6	-13.9	-9.2	10.5	0.7	0.0
4	5.9	2.3	-6.2	-3.0	-2.1	1.1
4.5	0.9	-1.3	-1.0	1.5	-	-
2006						
1.5	-	-	-	-1.5	-	1.7
2	7.7	-2.1	-4.6	-4.7	1.5	5.5
2.5	-2.4	-0.3	1.5	-1.3	0.3	2.0
3	1.2	-5.5	-2.1	1.5	1.5	3.2
3.5	2.3	-6.8	0.0	4.4	-3.9	3.2
4	3.7	-2.7	0.6	1.6	-22.0	6.7
4.5	0.0	-1.1	0.0	1.2	-	-
2008						
1.5	-	-	-	-2.1	-	2.9
2	5.6	-2.2	-2.0	0.5	-0.6	0.5
2.5	2.9	4.8	5.3	0.0	-9.0	-4.1
3	-2.4	-2.3	0.3	0.5	2.1	1.8
3.5	5.9	-5.8	-2.5	-1.5	-8.4	17.4
4	0.6	-2.5	3.3	1.5	-16.0	4.4
4.5	4.0	1.5	-3.1	-1.2	-	-

The table summarizes the differences of up- and down-grade probabilities between the momentum matrix and unconditional matrix for the years 2001, 2006 and 2008 according to the path dependent momentum matrices in Table 3.16 - 3.18. For example, up% Δ = upgrade probabilities of Up-Momentum-Matrix - upgrade probabilities of Unconditional Matrix. Figures are in %.

Table 3.21: Comparison of One-year Migration Matrices estimated with and without the Assumption of Time-homogeneity: 2001

Aalen-Johansen estimator (non-homogeneity)								
	1.5	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**59.9**	8.8	6.2	10.5	2.9	5.8	2.5	3.5
2	15.1	**45.6**	8.6	13.2	4.4	6.2	3.4	3.4
2.5	13.7	10.3	**39.4**	13.6	5.9	7.2	4.2	5.7
3	7.4	9.2	8.0	**45.9**	5.6	10.4	5.4	8.1
3.5	5.2	3.6	5.0	13.6	**43.7**	15.3	6.1	7.6
4	2.9	2.1	4.1	9.2	8.1	**52.0**	10.8	10.8
4.5	1.9	1.4	2.3	6.9	6.1	14.1	**58.1**	9.2
Duration (homogeneity)								
Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**61.1**	8.2	6.4	9.5	2.8	5.6	2.6	3.7
2	14.1	**48.0**	9.0	12.1	4.3	5.9	3.3	3.3
2.5	13.0	9.6	**42.2**	12.3	5.8	6.9	4.2	6.1
3	7.2	8.7	8.2	**46.9**	5.5	10.2	5.3	7.9
3.5	5.1	3.5	5.3	12.3	**45.8**	14.9	6.2	6.9
4	2.9	2.0	4.2	8.2	7.8	**54.0**	10.4	10.5
4.5	2.0	1.5	2.5	6.0	5.8	13.6	**60.2**	8.5
MT-Duration (homogeneity)								
Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**59.9**	8.5	6.6	9.9	2.9	5.8	2.6	3.7
2	14.8	**45.6**	9.5	12.7	4.5	6.1	3.5	3.3
2.5	13.5	9.9	**40.1**	12.8	6.0	7.1	4.4	6.1
3	7.5	8.9	8.6	**45.2**	5.7	10.5	5.5	8.1
3.5	5.3	3.6	5.5	12.9	**43.9**	15.4	6.4	7.0
4	2.9	2.1	4.4	8.6	8.2	**52.0**	10.8	11.0
4.5	2.0	1.5	2.6	6.3	6.1	14.2	**58.4**	8.8

The table compares the migration matrices based on continuous time setting between time non-homogeneous Aalen-Johansen estimator and time homogeneous estimators (Duration and MT-Duration) for 2001. The data covers the period from 1998 to 2009 for large loans over $1000k. The figures are in %.

Table 3.22: Comparison of One-year Migration Matrices estimated with and without the Assumption of Time-homogeneity: 2006

Aalen-Johansen estimator (non-homogeneity)								
	1.5	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**68.4**	11.2	6.2	4.0	3.0	2.6	2.1	2.5
2	21.1	**49.0**	9.9	7.2	4.0	3.1	2.3	3.5
2.5	18.7	12.2	**45.2**	8.3	4.6	4.4	2.9	3.7
3	13.3	10.3	12.5	**41.3**	6.9	5.3	4.2	6.3
3.5	8.8	9.1	9.1	11.6	**41.9**	7.9	5.3	6.2
4	10.3	7.2	6.0	8.2	7.9	**43.7**	9.2	7.5
4.5	6.0	5.4	4.8	5.5	5.9	8.8	**52.1**	11.4
Duration (homogeneity)								
Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**69.8**	10.6	5.9	3.9	2.9	2.5	2.0	2.4
2	20.0	**51.3**	9.3	7.0	3.9	3.0	2.3	3.3
2.5	17.2	11.3	**48.1**	8.2	4.5	4.3	2.8	3.5
3	12.4	9.6	11.7	**44.6**	6.7	5.1	4.0	5.9
3.5	8.1	8.3	8.4	11.2	**45.4**	7.6	5.0	6.0
4	9.5	6.4	5.7	8.2	7.7	**46.9**	8.8	6.9
4.5	5.5	4.9	4.4	5.3	5.7	8.5	**55.0**	10.8
MT-Duration (homogeneity)								
Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**68.7**	11.1	6.1	4.1	3.0	2.6	2.1	2.4
2	20.8	**49.3**	9.7	7.2	4.0	3.2	2.4	3.4
2.5	18.2	12.0	**45.4**	8.6	4.7	4.6	2.9	3.6
3	13.1	10.2	12.3	**41.7**	7.0	5.4	4.2	6.2
3.5	8.6	8.9	8.9	11.9	**42.2**	8.0	5.3	6.3
4	10.0	6.8	6.0	8.6	8.1	**43.9**	9.3	7.3
4.5	5.9	5.2	4.7	5.7	6.0	9.0	**51.8**	11.6

The table compares the migration matrices based on continuous time setting between time non-homogeneous Aalen-Johansen estimator and time homogeneous estimators (Duration and MT-Duration) for 2006. The data covers the period from 1998 to 2009 for large loans over $1000k. The figures are in %.

Table 3.23: Comparison of One-year Migration Matrices estimated with and without the Assumption of Time-homogeneity: 2008

Aalen-Johansen estimator (non-homogeneity)								
Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**70.0**	10.1	4.6	3.5	3.7	2.1	2.0	4.0
2	19.1	**51.2**	7.7	5.2	5.0	3.6	3.3	4.9
2.5	12.1	9.1	**47.4**	9.0	6.3	5.2	3.4	7.4
3	9.7	8.8	7.9	**47.9**	7.6	6.4	4.1	7.5
3.5	6.2	6.8	5.4	9.0	**48.4**	9.2	5.8	9.3
4	4.2	5.1	4.5	5.2	7.9	**52.7**	9.6	10.9
4.5	3.8	3.6	4.1	3.6	5.0	8.5	**55.7**	15.8
Duration (homogeneity)								
Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**71.0**	9.9	4.3	3.5	3.4	2.2	2.0	3.8
2	17.8	**53.7**	7.3	5.1	4.7	3.7	3.1	4.5
2.5	11.7	8.8	**49.7**	8.7	5.8	5.2	3.2	6.9
3	8.8	8.4	7.6	**50.5**	7.1	6.4	4.0	7.2
3.5	6.0	6.6	5.2	8.7	**50.2**	8.9	5.5	8.9
4	4.0	4.8	4.4	5.2	7.3	**54.5**	9.1	10.8
4.5	3.4	3.4	3.8	3.4	4.6	8.0	**57.8**	15.6
MT-Duration (homogeneity)								
Rating	*1.5*	*2*	*2.5*	*3*	*3.5*	*4*	*4.5*	*Default*
1.5	**69.8**	10.3	4.5	3.6	3.6	2.3	2.1	3.8
2	18.8	**51.4**	7.7	5.4	4.9	3.9	3.3	4.6
2.5	12.2	9.2	**47.5**	9.1	6.1	5.5	3.3	7.1
3	9.2	8.7	7.9	**48.2**	7.5	6.7	4.2	7.4
3.5	6.2	6.8	5.4	9.1	**48.2**	9.3	5.8	9.2
4	4.1	5.0	4.6	5.4	7.7	**52.5**	9.5	11.2
4.5	3.6	3.5	4.0	3.6	4.9	8.4	**55.5**	16.6

The table compares the migration matrices based on continuous time setting between time non-homogeneous Aalen-Johansen estimator and time homogeneous estimators (Duration and MT-Duration) for 2008. The data covers the period from 1998 to 2009 for large loans over \$1000k. The figures are in %.

Chapter 4

Forecasting Credit Migration Matrices with an Application to Macro Stress Testing

4.1 Introduction

Modeling and forecasting credit migration matrices conditional on macroeconomic conditions allows financial institutions to assess, analyze and manage the risk related to a credit portfolio. The Basel II and III Capital Accords require banks to conduct stress tests on credit portfolios under certain economic conditions that could have unfavorable effects on a bank's credit exposures in order to assess banks' ability to withstand abrupt market changes (Basel Committee on Banking Supervision, 2006). Under Basel II's credit risk framework, credit rating migration matrices are required to stress test marked-to-market (MTM) losses and related capital requirements(Gundlach, 2011). Subsequent to the melt-down of the U.S. subprime mortgage market, the global financial crisis urged implementation of stress-testing as a mandatory tool for credit risk management in many countries. In 2010, U.S. adopted the Dodd-Frank Act (DFA) which makes stress testing mandatory.

In the study of credit rating dynamics, the impact of business cycles on rating transitions is widely recognized. Using external rating data, Nickell et al. (2000) quantified the dependence of transition probabilities of bond ratings on the state of the business cycle. Bangia et al. (2002) showed significant differences in the loss distribution of credit portfolios by separating the economy into expansion and recession states. In terms of internal rating data, Krüger et al. (2005) illustrated that changes in migration matrices of an internal rating system could be related to macroeconomic factors using data from Deutsche Bundesbank. Mählmann (2006) found business cycle dependence of rating migration using a German bank portfolio with 85% SMEs borrowers. In Chapter 3, we also saw the relation between the time and migration matrices based on our SME loan portfolio.

Given the obvious importance of recognizing the impact of business cycles on rating transitions, the literature has generally used two techniques (Trück, 2008) to forecast migration conditional on macroeconomic indicators: the first class of approach uses factor models to build an unobserved credit indicator to model migration matrices under different (structural or reduced form) frameworks. Belkin et al. (1998), Kim (1999) used a one-factor structural model to forecast credit rating migration incorporating the business cycle through a systematic factor. Wei (2003) extended it to a two-factor model using the obligors' initial rating as a factor. Figlewski et al. (2012) incorporated simultaneous firm-specific and macroeconomic variables in a reduced-form model. This paper explored how general economic conditions impact defaults and major credit rating changes (upgrades and downgrades) and found that credit events are strongly influenced by ratings-related factors and are also significantly affected by macroeconomic factors. With exogenous and endogenous covariates, Bae and Kulperger (2009) and Berteloot et al. (2013) utilized an ordinal logistic regression model to predict robust migration matrices.

Another major technique for modeling migration matrix involves numerical adjustment methods (Jarrow et al., 1997; Lando, 2000). Given estimates for conditional default probabilities based on the macro-economic situation, a risk premium is chosen to adjust the average migration matrix in a way that the last column entries of the adjusted migration matrix matches the conditional default probabilities. Though the numerical adjustment method is convenient and is used by financial institutions, its accuracy has been questioned in Trück (2008).

Two other points should be stressed at this juncture:

- Existing papers used data from external rating agencies;
- Only Trück (2008) provided out-of-sample results of predicted migration matrices conditional on macroeconomic variables.

A rating migration model can be used for stress testing by simulating the impact of an economic downturn scenario on a credit portfolio. In the research of Tsaig et al. (2010), credit migration was found to explain as much as 51% of volatility and 35% of economic capital. And the migration of point-in-time credit quality accounts for a greater fraction of total portfolio risk when compared with through-the-cycle dynamics. Despite the broadly recognized importance of migration loss in stress tests, the literature on this issue is sparse. Most stress testing attempts focus on PDs (probabilities of default) (Carling et al., 2007; Eglemann and Hayden, 2008; Simons and Rolwes, 2009; Assouan, 2012; Dunbar, 2012). There are only two known studies which explicitly link business cycles to rating transitions with stress-testing applications. Miu and Ozdemir (2008) modify the one-factor model in line with Basel II requirements to stress test migration rates by conditioning the long-term migration matrix on macroeconomic variables. However, the benchmarking of the correlation parameters in the modified one-factor model poses a challenge in applications. Another paper taking account of changes in obligors' creditworthiness over the business cycle is Otani et al. (2009), in which macro stress-testing was performed on a Japanese bank's loan portfolio. The framework modified the multi-factor model of Wei (2003) to extract the common component of migration changes. The relationship between the common component and macroeconomic variables is examined in this context.

With the internal rating data of a Canadian SME loan portfolio, this chapter aims to (1) find the most influential macroeconomic variables for SME loans' default events; (2) forecast credit migration matrices reflecting business cycles; (3) examine the model performance of in-sample and out-of-sample periods via both mobility-based and risk-sensitive metrics; (4) apply the forecasting model to the assessment of credit migration loss and the stressed migration matrix; and (5) pursue these objectives using the Cohort and the Duration migration estimates respectively. In line with the Basel II credit risk model, we use a one-factor model (Kim, 1999; Trück, 2008) in which both PDs and migration rates can be stressed within a single framework. Another advantage of the one-factor model is to avoid redundant modeling development. If one bank already has an in-house PD model incorporating macroeconomic variables, it could use the existing PD model in the one-factor model framework. Using *Manufacturing* large loans data from our SME portfolio covering from 1998Q1 to 2009Q4, we calibrate the one-factor model in two steps: building a credit cycle index and conditioning the transition matrices. In the first step, 26 macroeconomic variables from four categories (Financial Markets, Stock Markets, National Accounts and Composite Indices) are used to select the significant covariates on which to build a PD-based credit cycle index to represent the credit state of the financial market. In the second step, migration matrices are calculated conditional on the credit cycle index and subsequently compared with benchmarking matrices via risk-adjusted distance metrics (Trück, 2008) and a mobility-based metric (Lando and Skodeberg, 2002). With the predicted migration matrices, we demonstrate to what degree the migration matrices impact the portfolio credit value and related credit losses. A stressed migration matrix is constructed using the one-factor model under a hypothetical stress scenario.

The remainder of this chapter is structured as follows. The methodology of the general framework and the one-factor model are presented in the next section. Section 3 describes the SME data and macroeconomic variables that are used to condition migration matrices on the business cycle. Section 4 calibrates the model and provides empirical results on the in-sample and out-of-sample migration matrices. The application of predicted migration matrices on portfolio credit value change and stress tests are examined in section 5, and section 6 concludes.

4.2 Modeling Methodology

4.2.1 General Framework

There are two classes of models for credit rating change in the literature: structural models and intensity-based models. As discussed in Gagliardini and Gourieoux (2005), the two approaches are special cases of ordered polytomous models with different assumptions on the error term. The idea of an ordered polytomous model is to introduce an unobserved quantitative indicator from which an obligor's credit quality is determined. We briefly summarize this approach.

In a finite state space $S = \{1, 2, \dots, K\}$, we denote credit rating at date t for an obligor r by $R_{r,t} \in S$. $Y_{r,t}$ represents the underlying quantitative indicator, or credit change indicator for obligor r at date t. Assuming the conditional distribution of $Y_{r,t}$ depends on a factor Z_t (which can be multidimensional) and the previous rating $R_{r,t-1} = i$, we have:

$$Y_{r,t} = \beta_i Z_t + \sigma_i u_{r,t}, \tag{4.2.1}$$

where $u_{r,t}$ are i.i.d. variables with cumulative distribution function G, and independent of common factor Z_t. Let us assume that the quantitative indicator $Y_{r,t}$ in 4.2.1 determines the credit rating $R_{i,t} \in S$ at date t through the relation:

$$R_{r,t} = j, \text{iff} c_{j-1} \leq Y_{r,t} \leq c_j, \tag{4.2.2}$$

where $-\infty = c_0 < c_1 < \dots < c_{K-1} < c_K = \infty$ are thresholds which are fixed but unknown to the public. Then the transition probability $p_{r,t}(i,j)$ for obligor r from state i to j at time t is given by

$$\begin{aligned} p_{r,t}(i,j) &= P[R_{r,t} = j \mid R_{r,t-1} = i, Z_t] \\ &= P[c_{j-1} \leq Y_{r,t} \leq c_j \mid R_{r,t-1} = i, Z_t] \\ &= P[c_{j-1} \leq \beta_i Z_t + \sigma_i \epsilon_{r,t} \leq c_j \mid R_{r,t-1} = i, Z_t] \\ &= G(\frac{c^i_j - \beta_i Z_t}{\sigma_i}) - G(\frac{c^i_{j-1} - \beta_i Z_t}{\sigma_i}). \end{aligned} \tag{4.2.3}$$

When the error terms $u_{r,t}$ are normally distributed, the model is called a probit model, also referred to as a structural model in the literature (Kim, 1999; Nickell et al., 2000; Trück, 2008); When the error terms $u_{r,t}$ have the Gompertz distribution $G(x) = 1 - exp[-exp(x)]$, the model is called a Gompit model which is generally called a reduced form model (Figlewski et al., 2012); when the error terms have a standard logistic distribution $G(x) = \frac{exp(x)}{1+exp(x)}$, the model is a logistic model (Bae and Kulperger, 2009; Berteloot et al., 2013), which is often used in industry practice for modeling the probability of default.

Based on the above general model framework, there have been different extensions and modifications of the credit change indicator (equation 4.2.1) in modeling migration matrices. Kim (1999), Trück (2008) and Miu and Ozdemir (2008) described the credit change indicator Y_t via a Vasicek one-factor model, which is commonly used in the financial industry in the Basel II framework. Macroeconomic variables are included to build the single factor Z_t. To accommodate heterogeneity in obligors, Bae and Kulperger (2009) and Figlewski et al. (2012) incorporate firm-specific covariates besides macroeconomic variables to describe the credit change indicator with a multidimensional vector Z_t. Instead of using exogenous variables, Belkin et al. (1998) and Gagliardini and Gourieoux (2005) constructed the factor Z_t through a Gaussian (vector) autoregressive process.

4.2.2 One-factor Probit Model

In line with the Basal II requirement, we follow Kim (1999) and Trück (2008) to use the one-factor probit model to forecast conditional migration matrices using macroeconomic variables. We assume that an underlying continuous credit-change indicator Y_t conditional on the previous rating category, following a standard normal distribution, is influenced by a systematic risk factor Z_t and idiosyncratic risk component ϵ_t:

$$Y_t = \omega_i Z_t + \sqrt{1-\omega_i^2}\epsilon_t, \tag{4.2.4}$$

where Z_t and ϵ_t are assumed to follow mutually independent standard normal distributions. The parameter w represents the correlation between Z_t and Y_t. Here Z_t could be regarded as a credit cycle index which indicates the credit state of the financial market as a whole. Note that we suppress the subscript r for obligor in the equation from general form 4.2.1 with the assumption that obligors having the same rating category follow the same credit change indicator Y_t. The probability distribution for the rating change of an obligor then relates to the outcome of the credit cycle index Z_t, which is designed to be positive on good days and to be negative on bad days.

The conditional transition probabilities from rating i to rating j is:

$$p_t(i,j|Z_t) = \Phi\left(\frac{c_j^i - \omega_i Z_t}{\sqrt{1-\omega_i^2}}\right) - \Phi\left(\frac{c_{j-1}^i - \omega_i Z_t}{\sqrt{1-\omega_i^2}}\right). \tag{4.2.5}$$

Now we discuss the parameters' calibration in the one-factor model, including the credit cycle index Z_t, threshold c, and weight w.

Systematic Factor A simple way to construct Z_t is to use the probability of default (PD) as a proxy for the credit cycle index. Since PDs of higher quality rated bonds are rather insensitive to the economic state (Wilson, 1997; Belkin et al., 1998), CreditMetrics Model and Kim (1999) used the PDs of speculative grade bonds (rated equal to and lower than Moody's Ba rating) while Trück (2008) built two Z_t indices with PDs of speculative and investment grades bonds. Now let S_t be the default probability of speculative loans in period t. μ_s and σ_s are denoted as the mean and the standard deviation of the inverse normal transformation $\Phi^{-1}(S_t)$ of S_t, respectively. Thus the credit cycle index Z_t

$$Z_t = \frac{\Phi^{-1}(S_t) - \mu_s}{\sigma_s} \tag{4.2.6}$$

follows a Guassian distribution of expectation 0 and standard deviation of 1.

We can then model the default probability S_t as a probit model:

$$S_t = \Phi(\beta X_{t-1} + \varepsilon_t), \tag{4.2.7}$$

where Φ denotes the cumulative standard normal distribution function, X_{t-1} is a set of macroeconomic variables and ε_t is a random error term with $E_{t-1}(\varepsilon_t) = 0$. The inverse normal CDF transformation converts equation 4.2.7 to a linear regression as

$$\Phi^{-1}(S_t) = \beta X_{t-1} + \varepsilon_t. \tag{4.2.8}$$

and the coefficients β can be estimated via OLS. Then the forecasted inverse normal CDF of the speculative PD is:

$$E_{t-1}(\Phi^{-1}(S_t)) = \hat{\beta} X_{t-1}, \tag{4.2.9}$$

where $\hat{\beta}$ are the estimated coefficients.

Thresholds The threshold $\bar{c}^i_j$ for a transition from rating grade i to grade j is determined by historical average transition rates: $\bar{c}^i_j = \Phi^{-1}(\sum_{k=j}^K \bar{p}_{ik})$. The thresholds effectively divide the real line into disjoint intervals corresponding to rating changes. Then the credit-change indicator Y_t will determine a rating change depending on its bin.

Weights The credit-change indicator Y_t is determined from the credit cycle index Z_t and ω. Given Z_t, we calibrate ω so that the probability structure governing Y_t (equation 4.2.5) approximates the thresholds structure based on historical average transition rates just defined. Accordingly, the weight ω in equation 4.2.5 is determined so as to minimize the difference between the model transition probabilities and the observed transition probabilities.

There are three major methods to characterize the distance measure between two matrices: traditional matrix norms, mobility-based metrics and risk-adjusted difference indices. Traditional matrix norms are classical cell-by-cell distance measures. Though these intuitive measures were widely used to compare migration matrices in early research (Belkin et al., 1998; Israel et al., 2001; Bangia et al., 2002; Wei, 2003), they are not optimal to measure changes in migration matrices (Trueck and Rachev, 2009, chapter.7). The mobility-based metric proposed by Jafry and Schuermann (2004) and the risk-sensitive difference indices introduced by Trueck and Rachev (2005) are more often used in recent literature (Lando and Skodeberg, 2002; Xing et al., 2010; Trück, 2008; Berteloot et al., 2013).

In the estimation of the weight w, we will use both mobility-based and risk-sensitive metrics to minimize the distances between the modeled and empirical transition probabilities. The definitions of these two types of metrics are described in Chapter 3.

To fit our data within this modeling framework, we proceed in 5 steps in the following sections:

(a) Determination of transition horizon and segmentation of the rating data in Section 4.3.1.

(b) Determination of macroeconomic variables X_{t-1} needed in modeling the probabilities of default are explained in Section 4.3.2. Their correlation and detrending procedures are discussed in Section 4.4.1.

(c) Regression calibration to forecast probabilities of default is undertaken in Section 4.4.1, as well as the construction of the credit cycle index Z_t.

(d) Section 4.4.2 deals with the calculation of the threshold $\bar{c}^i_j$ in and the estimation of the weight ω.

(e) Section 4.4.2 describes the construction of one-period ahead forecast of the conditional migration matrices.

4.3 Rating Data and Explanatory Variables

4.3.1 Rating Data

The data used in this chapter comes from the internal ratings of *Financing Company* whose rating system and data are described in Chapter 2. The SME loan portfolio covers 12 years of rating history from 1998 to 2009, which is divided into three subsamples via loan size threshold: small loans (\$50k-\$250k), medium loans (\$250k-\$1000k) and large loans (above \$1000k). A seven-position numerical rating system from 1.5 to 4.5 is constructed to reflect credit quality deterioration.

The empirical migration matrices used in this chapter are calculated through both the Cohort method and the Duration method. As we discussed in Chapter 3, the differences of migration matrix estimates between the Cohort method and the intensity-based estimates are larger than the differences between the Duration and MT-Duration estimates. Considering the computational cost, we use the Duration estimates to investigate the implementation effect of intensity-based migration estimates in the forecast modeling of migration matrices. The calculation procedure for the two methods are described in Chapter 3.

Now we need to determine the appropriate transition period to study the effect of macroeconomic variables on migration matrices. Naturally, we expect longer transition periods (i.e., one year) to yield more stable transition probabilities. However, there are drawbacks associated with using too long a transition period. First, the number of observed migration matrices will be relatively small under a yearly transition period. In our case, there will be only 12 data points which is not enough to have sound results for regression analysis. Second, the lost information within the yearly transition period increases, since all rating changes within the sample are not taken into account via the Cohort method. Accordingly, we use a quarterly transition period in this chapter.

We consider two filters to segment the rating data for modeling transition probabilities conditional on macroeconomic variables. First, *Manufacturing* sector data are applied to model conditional migration matrices. In examining latent risk factors in migration modeling, Wendin and Mcneil (2006) found that it is imperative to address the issue of heterogeneity among industry sectors. As well, using the broad-based SME whole sample data, we could not calibrate the model with acceptable regression performance. In order to reduce the heterogeneity of data, we utilize the sectoral data (Gagliardini and Gourieoux, 2005) to estimate the model. As we discussed in Chapter 2, *Manufacturing* is the largest sector accounting for over a quarter of the portfolio in terms of obligor years. As well, more than half of obligors in *Manufacturing* are large loans. Second, large loans (above $1000k) data are used in the modeling. We discussed in Chapter 3 that the *Financing Company* changed the policy of rating review for loans below $1000k after 2005. This fact led to a structure break of sequential migration matrix for small and medium loans as shown in Figure 4.1 - 4.2. To avoid the effect of policy change of the rating review process, this chapter only uses large loans (above $1000k).

In all, the final sample data contains 221,368 obligor quarters with 1,918 defaulted and 14,672 non-defaulted distinct obligors.

4.3.2 Macroeconomic Variables

With regard to the macroeconomic variables, we choose 26 macroeconomic and financial series; these are shown in Table 4.1. They are grouped into four categories: Financial Markets, Stock Markets, National Accounts and Composite Indices from various resources including Statistics Canada, Bloomberg, and the St. Louis FRED Economic Data.

We have tried to include variables that have been explored in previous research, such as the unemployment rate, inflation, GDP growth, stock returns etc. (Kim, 1999; Figlewski et al., 2012; Simons and Rolwes, 2009). In addition to the obvious candidates, we examined a broad selection of macro variables that are tailored for SME *Manufacturing* data: Production in manufacturing, Producer price index in manufacturing, Purchasing manager index in manufacturing; two variables are for small and medium firms: the Russell 2000 Index and the Spread of Loan Rate over Cost of funds for Small Firms. Corresponding to the loan portfolio data, we use quarterly data for macroeconomic variables.

Financial Markets To measure the extent of raising capital for borrowers, we include interest rates, relevant spreads and exchange rates.

- *Interest rates*: 3-month T-bill rate, 5-year and 10-year Treasury yield and Bank interest rate. As in other studies, we use the Canadian 3-month treasury bill yield as a measure of the tightness of money market and Canadian notes yield of 5-year and 10-year as a measure of the overall level of interest rates at longer maturity. Meanwhile, central bank (Bank of Canada)'s administered bank rate is also included for consideration. Duffie et al. (2007) found that short-term interest rates were negatively related with credit default intensity, which is consistent with the fact that short rates are often increased by central banks to cool down business expansion. While short-term interest rates are administered by central banks, long-term interest rates are determined by market forces. The higher the yield for a 10-year note or a long term bond, the more optimistic markets are about the economy.

- *Spreads*: government bond yields spread, corporate bond yield spread, and the spread of loan rate over banks' cost of funds. The slope of the yield curve is one of the most powerful predictors of future economic growth. Many studies (Keenan et al., 1999; Duffie et al., 2007; Trück, 2008) have underscored its importance in modeling the probability of default. We follow the definition of yield curve slope in (Estrella and Hardouvelis, 1991) to include the spread between Canadian 10-year government bond rate and 3-month T-bill rate. Similarly, we also include the spread between the Canadian 5-year government bond rate and 3-month T-bill rate. Meanwhile, from the viewpoint of the corporate sector, we consider corporate credit spreads on high-yield bonds as explanatory variables. As discussed in Chapter 2, SME loans of the *Financing Company* generally correspond to the BB broad rating class of external rating system. Accordingly, we include the spread between corporate Baa yields and constant maturity 10-year Treasuries see (Figlewski et al., 2012); and the spread between Aaa and Baa corporate bond yield (Kim, 1999). In addition, we include the spread of loan rates over banks' cost of funds for small firms in order to cater to SME data.

- *Exchange rate*: the obligors that have a great deal of international business are expected to be affected by exchange rates. However, the sign of the relation varies according to the business type. Importing businesses are positively affected if the exchange rate is high; exporting businesses are negatively affected.

Stock Markets In structural form default risk theory (i.e., Merton's model), the probability of default is negatively related to stock market returns and positively related to volatility.

- *Stock Index return*: S&P/TSX index, and Russell 2000 index. We include S&P/TSX (Toronto Stock Exchange) Composite Index return to measure the general health of the corporate sector in Canada. However, the S&P/TSX index is based on the market capitalizations of large companies. To examine smaller companies' performance in stock market, we also include the return of the Russell 2000 index, which is a small-cap stock market index of the bottom 2,000 stocks in the Russell 3000 Index.

- *Volatility Index*: as a widely used measure of market risk, the CBOE volatility index is designed to measure the expected 30-day volatility for the S&P 500. It is constructed using the implied volatilities of a wide range of S&P 500 index options.

- *Price Earnings Ratio*: a standard way to investigate market valuation is to study the historic Price-to-Earnings (P/E) ratio. We include Shiller P/E10 Ratio, which is based on the real monthly averages of daily closing prices of the S&P 500 divided by the average inflation-adjusted earnings from the previous 10 years.

National Accounts A variety of broad economic national activities might influence corporate credit risk. We accordingly choose several economic indicators.

- *Real GDP*: Gross Domestic Production (GDP) represents aggregate demand of an economy which relates to the sales of firms. Lower GDP growth might influence the income of firms, which increases the possibility that firms cannot meet obligation and default accordingly.

- *Unemployment rate*: the unemployment rate is a typical indicator of the overall health of the economy. In industry practice and the academic literature, it is quite commonly used in default risk modeling.

- *CPI Inflation*: Canada consumer price index, which measures the changes in the cost of a fixed "basket" of consumer purchases, is included to consider the influence of inflation on default risk. An increase in the CPI suggests inflation. As argued in Figlewski et al. (2012), the relation between inflation and default risk has two possible results. The common perception is high inflation should increase default risk; however, inflation could reduce the real value of outstanding debt in terms of nominal dollars, which might make defaults less likely.

- *Production*: to capture aspects of production activities in Canada, we include the Industry production price, Industrial production and capacity utilization. Considering the sample data used in this chapter, we also choose related variables specialized in *Manufacturing* sector, including manufacturing production, manufacturing producer prices index, and manufacturing purchasing manager index (PMI). Specifically, capacity utilization measures how close current production is to maximum capacity; PMI index is an indicator of the economic health of the manufacturing sector base on five major aspects: new orders, inventory levels, production, supplier deliveries and the employment environment. Higher values correspond to a stronger economy.

Composite Indices Previous studies indicated that some comprehensive index is highly related to corporate default risk. We therefore include Chicago Fed National Activity Index

(CFNAI) and St. Louis Financial Stress Index (STLFSI).

- *CFNAI*: aiming to gauge overall economic activity and related inflationary pressure in a single index, the Chicago Federal Reserve publishes the CFNAI, which summarizes the behavior of 85 economic series in four major groups: production and income; employment, unempolyment and hours; personal consumptions and housing; and sales, orders and inventories. Several previous studies (Stefanescu et al., 2009; Figlewski et al., 2012) included this composite index which exhibited high explanatory power on default risk.

- *STLFSI*: in early 2010, the St. Louis Fed created the STLFSI from seven interest rate series, six yield spreads and five other indicators in an attempt to avoid focusing on a single indicator at the expense of others. The average value of the index, which begins in late 1993, is designed to be zero representing normal financial market conditions. Values below zero are a sign of below-average financial market stress, while values above zero suggest above-average financial market stress.

4.4 Calibration and Empirical Results

4.4.1 Building the Credit Cycle Index

In the first step, a credit cycle index Z_t is constructed through speculative probabilities of default via equation 4.2.6. Using the SMEs loan portfolio, the internal rating system has the top credit quality rating equivalent to the speculative grade of external ratings according to the magnitude of PDs. Accordingly, the PDs of our sample data is thought to be sensitive to the economic situation and will be used to build the index Z_t. It is implicit that we are effectively describing two approaches in so far as the value of empirical PDs are model dependent (Cohort and Duration).

Now the question is how to calibrate the regression as shown in equation 4.2.8 to forecast future PDs (more exactly, the inverse normal CDF of PDs). Using in-sample period empirical PDs and the macroeconomic covariates from 1998Q1 to 2007Q4, we will examine variable data, estimate the parameters of regression and test regression assumptions.

4.4.1.1 Variable Determination

Before we move to the estimation of the regression, we need to obtain stationary macroeconomic data and examine the correlations among macroeconomic variables.

First, macroeconomic time series data normally are not stationary. In macroeconomics, the Hodrick-Prescott (HP) filter is the standard technique for separating the long run trend in a data series from short run fluctuations (Kydland and Prescott, 1990). We follow the literature and use the HP filter to detrend the macroeconomic variables. Table 4.2 exhibits the statistics of those cyclical components of macroeconomic series data that will be used in the modeling.

Second, the occurrence of highly correlated covariates could lead to issues in multicollinearity. One remedy is to eliminate variables that are highly correlated with others. As shown in Table 4.3, most of the correlations are quite moderate, with higher values where expected. For example, the two bond yields variables ($GCAN10YR$ and $GCAN5YR$) have a correlation of 0.92; the two spreads of bond yield variables accordingly ($SP10\_3$ and $SP5\_3$) have a correlation of 0.97; and two interest rate variables ($BINTR$ and $GCAN3M$) have a correlation of 0.97. Meanwhile, we also see high correlations between the unemployment rate ($UNEMR$) and the three production variables ($IPROD, CAPUTL$, and $MPROD$). Those variables are also highly correlated with each other, as indicated by the bold figures. However, all of them have been found to be important in earlier research, so we keep them for now in the specification. We have eliminated two variables: $BINTR$ and $SP5\_3$.

4.4.1.2 Regression Calibration

The goal is to select and estimate the variables in equation 4.2.8. We should keep in mind the default probabilities are model dependent and we will undertake the selection process for probabilities determined by the Cohort and the Duration methods.

In the variables selection of the regression model, we follow Figlewski et al. (2012) to use a stepwise backward selection procedure with all macroeconomic variables to eliminate those that were not statistically significant. Starting with all variables included, the one variable with the least significant p-value will be eliminated from the specification and the model is refitted. This process is repeated until all remaining coefficients are significant at at least the 5% level. The results are shown in Table 4.4. The macroeconomic variables that have survived the backward selection process are similar to those in the final specification using either the Cohort and the Duration migration matrices. The five variables based on the Cohort estimates are *Manufacturing production* ($MPROD$), *St. Louis financial stress index* ($SLFSI$), *Canadian 5 year government bond* ($GCAN5YR$), *Canadian dollar effective exchange rate index* ($CAEXR$), and *SPX volatility index* (VIX). Four macroeconomic variables based on the Duration estimates include the same first three variables as those in the Cohort specification plus *Spreads of loan rates over banks' cost to small firms* ($SPLR\_BCF$).

Within Financial Markets, three variables ($GCAN5YR$, $CAEXR$ and $SPLR\_BCF$) are significant in the regression models. In a weak economy, the increasing demand on long-term treasury bonds creates a low bond yield. Accordingly, we observe a negative relation between the default probabilities and $GCAN5YR$. The negative coefficient of $CAEXR$ indicates that high exchange rates decrease the default risk of SME loans. In addition, we see that neither short term interest rates nor credit spreads are retained in the backward selection procedure, which contrasts with the results of existing research (Kim, 1999; Trück, 2008; Figlewski et al., 2012) based on external rating data.

Within Stock Markets, only S&P VIX survives the selection process, albeit with an anomalous negative sign: a volatile stock market for large firms is associated with decreased SME default risk. In Duffie et al. (2007) and Figlewski et al. (2012), it was S&P 500 returns instead of volatility that survived the variable selection. However, both of them reported an odd positive sign between S&P 500 returns and default intensity. As argued in Duffie et al. (2007), this result could be due to correlation between VIX and other variables, and perhaps due to the nature of SME default events and business-cycle dynamics.

Turning to National Accounts, we select the sectoral variable of $MPROD$ rather than the usual variables such as GDP growth, Unemployment rate, and CPI that are selected by regressions using external rating data. With an odd positive sign, $MPROD$ highly influences *Manufacturing* SME default risk. A strong manufacturing market appears to be a warning signal for the SME credit market.

Finally, $SLFSI$ has a statistically significant positive coefficient in the specification of the regression model. When financial market stress increases, the SME default risk increases accordingly.

Table 4.4 provides the values for the estimated coefficients ("Coef") on the variables in their natural units. To aid in interpreting the impact of each variable, we use metric-free standardized coefficient values ("Std Coef"), which are obtained by multiplying each raw coefficient by the ratio of standard deviations of the independent variables (see Table 4.2) and the dependent variable, to better gauge the effective relative importance of these variables. It measures the effect on the dependent variable ($\Phi^{-1}(S_t)$) when an independent variable increases by one standard deviation. For example, consider the most influential variable $SLFSI_{t-1}$ in the Cohort specification: increasing one standard deviation of $SLFSI_{t-1}$ will increase on average 1.02 standard deviations of $\Phi^{-1}(S_t)$. According to the standardized coefficients, the most important three variables are $SLFSI_{t-1}$ (St. Louis Stress Index), $CAEXR_{t-1}$ (exchange rate) and $MPROD_{t-1}$ (manufacturing production) respectively for the Cohort specification and $SLFSI_{t-1}$(St. Louis Stress Index), $SPLR\_BCF_{t-1}$ (spread of loan rate over cost for small firm) and $MPROD_{t-1}$ (manufacturing production) for the Duration specification.

The adjusted R-squared statistics given by the models using the Cohort and the Duration migration estimates are 0.65 and 0.44 respectively, which are much lower than the results (0.98) in Trück (2008) using Moody's speculative grade data. In line with the discussion of retained variables above, the default risk of the SME loan portfolio seems be less sensitive to general economic conditions than that of large corporations. However, the models are significant with F-statistics (16.05 for the Cohort specification and 8.97 for the Duration specification) having p-values close to 0.

Figures 4.3 - 4.4 show the regression performance for the in-sample period (1998Q1-2007Q4) and the out-of-sample period (2008Q1-2009Q4); out-of-sample values are determined by the fixed regression coefficients determined in-sample and out-of-sample values of the external variables. In general, the model captures the cycles of default risk for *Manufacturing*, especially the changing direction in the out-of-sample period. Compared to the empirical data, we could see in Table 4.5 that three out of eight predicted values have high accuracy: 2008Q3, 2008Q4 and 2009Q2 for the results using Cohort migration and 2008Q3, 2008Q4 and 2009Q3 for the results using Duration migration.

We also conduct certain diagnostic tests. Using studentized residuals, there is no evidence of autoregressive correlation and heteroskedasicity. The normality assumption of residuals is nearly satisfied and multicollinearity is not a issue in the regression. These results hold for both Cohort and Duration specifications and are available on request.

With the fitted value of $\Phi^{-1}(\hat{S}_t)$, we can build the estimated credit cycle index $\hat{Z}_t$.

4.4.2 Constructing Transition Matrices

We now proceed to the second step in constructing a migration matrix. Two parameters need to be calibrated: thresholds $\bar{c}^i_j$ and weight ω_i.

Our internal rating data contains 8 ratings categories including default. Using historical average transition rates for the in-sample period from 1998Q1 to 2007Q4, we determine 7 threshold values to define the bins as shown in Table 4.6 using the Cohort estimates of migration matrices. Taking transition from grade 3 as example, we observe a 168 bps default rate. Using the inverse cumulative probability function for a standard normal distribution, $c^3_D = \Phi^{-1}(p_{3,D}) = \Phi^{-1}(0.0168) = -2.12$. The upper threshold of the default bin is -2.12. Next for the bin of grade 4.5, we obtain $c^3_{4.5} = \Phi^{-1}(p_{3,D} + p_{3,4.5}) = \Phi^{-1}(0.0168 + 0.017) = -1.82$. The calculation of the other thresholds and bins follows the same procedure. Similarly, the thresholds and bins using the Duration migration matrices are presented in Table 4.7.

The parameters ω_i should be estimated for each credit rating to reflect the different sensitivity of each credit rating to the credit cycle index Z_t. As discussed in Kim (1999), this estimation method has overfitting or data-snooping problems because of the small sample size for each rating grade. Instead, the paper estimated the parameter for two pools of credit ratings - the investment grade and speculative grade - with stable results. Since the data used in this chapter is segmented both by size (large sample) and sector (*Manufacturing*), the small sample size for each rating is indeed a concern. In the spirit of Kim (1999), we will use the whole sample to estimate ω reflecting the speculative nature of the SME data.

The estimation problem now lies in finding the best ω to minimize the difference between the conditional forecast $p_t(i, j|\hat{Y}_t)$ and empirically observed migrations $p_t(i, j)$ for each time period t. Given the discussion in Section 4.2.2, the three criteria are the risk-sensitive metrics (D1, D2) and the mobility-based metric SVD. Accordingly, the credit change indicator $\hat{Y}_t$ is then an outcome of the forecasted credit cycle index of the next period ($\hat{Z}_t$) and the estimated weight $\hat{\omega}$. The estimation of weights will be conducted with in-sample and out-of-sample data respectively in the next two subsections.

4.4.2.1 In-sample Estimates of the Conditional Migration Matrices

With the parameters - coefficients $\hat{\beta}$, the thresholds c and the weight $\hat{\omega}$ - calibrated from the in-sample period (1998Q1 - 2007Q4) data, we will obtain one-period ahead estimates of conditional migration matrices for each quarter of the in-sample period.

We first consider the results of in-sample estimated weight $\hat{\omega}$ in equation 4.2.5. Table 4.8 provides the weight giving the minimal distance between the estimated and empirical migration matrices for the in-sample period of 1998 - 2007. Depending on the chosen distance measures, we obtain different outcomes for the weight. For both the Cohort and Duration estimates, our results based on the risk-sensitive metrics are in line with those (0.21-0.23) for the speculative grade in Trück (2008) even though we have relatively lower weights. The internal rating data of SME loan portfolio appears be less sensitive to systematic influence compared with the external rating data.

We now investigate the in-sample one-period ahead estimation results for the different approaches. For each period t, we use the value of the macroeconomic variables at $t-1$ to determine the systematic factor Z_t. The conditional migration matrix for each period time is then obtained using the one-factor model. The parameters (coefficients $\hat{\beta}$, the thresholds c and the weight $\hat{\omega}$) used in this estimation are those calibrated over all 40 quarters of the in-sample period. The error between the estimated conditional migration matrix obtained in this fashion and the empirical one is calculated for each period t based on the chosen distance metric. Goodness-of-fit in this exercise is measured by determining the mean absolute error over the 40 quarters.

To investigate the credit cycle influence (one-factor model) on the estimates of the conditional migration matrices, two standard benchmark unconditional methods are used: the average migration matrix over all in-sample periods (Naive I); and the transition matrix of the previous period (Naive II). Both models ignore the credit cycle in different ways.

Table 4.9 provides in-sample results for mean absolute errors (MAE) of estimates according to the applied different metrics (D1, D2 and SVD) for the one-factor conditional approach (Model), Naive I and Naive II. Best results for each distance measure are highlighted in bold. Comparing MAE within the same distance measure (in the rows), we can see that the one-factor model outperforms the two naive approaches in both mobility-based and risk-sensitive metrics. With the metrics of D1 and D2, the one-factor model provides 30-40% less MAE for the Cohort migration matrices and 50-60% less for the Duration migration matrices; With the SVD metric, the one-factor model is still superior to the benchmarks but only provides 10-20% less MAE. It is worth noting that the mean error or the standard deviation of the errors for different indices within the columns cannot be compared because of different scales. However, the results in the rows can be used to compare forecasting performance.

We could conclude that the one-factor model provides the best estimates of conditional migration matrices since it incorporates more business cycle information via the systematic factor Z_t than the other two benchmark methods.

4.4.2.2 Out-of-sample Forecasts

Finally, we use the one-factor model to forecast rating migration behavior for the out-of-sample period from 2008Q1 to 2009Q4. The in-sample estimation period is increased each quarter from $1998Q1-2007Q4$ to $1998Q1-2008Q1, 1998Q1-2008Q2, \cdots, 1998Q1-2009Q4$. Using a quarterly re-estimation of the bins, the weight of the systematic factor and the credit cycle index Z_t, the one period ahead forecasts of conditional transition matrix for each quarter during the out-of-sample period (2008Q1-2009Q4) is determined. In contrast with the in-sample estimated parameters, out-of-sample forecasts are based on parameters derived from current information.

Table 4.10 exhibits the re-estimation of weights for the out-of-sample period. The weights increase over time for all metrics and both migration estimates, varying from 0.09 to 0.22 for risk-sensitive metrics and from 0.13 to 0.18 for mobility-based metric. The out-of-sample period correspond to an economic downturn period caused by the financial crisis. It seems that SME credit migration behaviors could be explained more by the systematic factor in an adverse situation. Similarly in the in-sample period, the out-of-sample weights from the Duration migration are still larger than those from the Cohort migration.

We notice that the $SLFSI$ at 2008Q4 is extremely high as six standard deviations of history since 1998Q1, which drive the regression result of the credit cycle index. Considering $SLFSI$ is a composite index, we use its three standard deviation value in out-of-sample forecast to better describe default behavior of our data.

Out-of sample forecast performances of difference approaches are provided in Tables 4.11 - 4.12. The error between the estimated conditional migration matrix obtained in this fashion and the empirical one is calculated for each period t based on the chosen distance metric.

The results are not as strikingly in favor of the factor model as they were in the in-sample results. With risk-sensitive metrics D1 and D2, the factor model outperforms the other two benchmarks only for those time points, when we have a good estimate of PD. These are 2008Q3-2009Q2 for the Cohort migration and 2008Q3-2009Q1 and 2009Q3 for the Duration migration (see Table 4.5).

At other points, the Naive I approach (historical average migration) performs better than the Naive II (previous migration) under the risk-sensitive criteria. This result provides some practical support for the industry practice that uses historical average migration matrices in the assessment of migration loss.

A last point concerns the mobility-based SVD. The results indicate that according to this metric, the one-factor model provides inferior forecasts compared to the two naive approaches. One possible explanation is that one-factor model is not appropriate. Another is that the SVD metric is itself an inappropriate measure to access difference in migration matrices. The distributions of portfolio credit value change using predicted migration matrices in the next section provide some empirical support for the latter explanation.

4.5 Application of Migration Matrices

In this section, we use the conditional migration matrices forecasted from the one-factor model to calculate portfolio credit value. In addition, a stressed migration matrix can be developed through the calibrated one-factor model to determine economic capital under the stress scenario is analyzed.

4.5.1 Portfolio Credit Value

Corporate credit risk, known as "wholesale credit risk", as opposed to "retail credit risk", requires us to consider not only default loss but also mark-to-market loss through the migration matrix. The current procedure in industry commonly uses the historical average migration matrix regardless of the changes to the migration matrix through the business cycle. Kim (1999) and Bangia et al. (2002) demonstrated how the portfolio value distribution is impacted by a time-varying transition matrix. To illustrate the extent to which a one-factor modelled migration matrix could improve the accuracy of the credit loss distribution, we adopt Mählmann (2006) to construct an artificial portfolio to compare the economic capital across the one-factor model, Naive I (historical average), Naive II (previous) and empirical transition matrices. Noted that the credit loss distribution here includes the mark-to-market migration loss and default loss, which in contrast to the one in Chapter 3 that only included default loss because the unavailability of SME loans' cash flow.

A one-factor CreditMetric model (see Gordy (2000) and Chapter 3 for details) is applied to assess portfolio capital revaluation and credit loss. Associated with obligor r is an unobserved latent variable X_r, which is driven by a systematic factor Z and an idiosyncratic component ϵ_r:

$$X_r = wZ + \sqrt{1-w^2}\epsilon_r, \tag{4.5.1}$$

where Z and ϵ_r follow independent standard normal distributions and factor loading w determines the relative importance of the idiosyncratic risk for the obligor. Under this framework, we will determine the capital revaluation that reflects not only changes in defaults, but also rating migrations using the following steps:

- The correlated latent values X_r are generated by simulation of two independent standard normal distributions Z and ϵ_r via equation 4.5.1.
- Assuming a normal distribution of ratings, the cutoff C_j^i associated with rating i is

obtained by $C_j^i = \Phi^{-1}(\sum_{l=j}^{K} p_{il})$, where p is the transition probability.

- The state of obligor i at the risk-horizon depends on the location of X_r relative to the set of "cut-off" values determined in the last step.
- The asset value of obligor i at the risk-horizon is obtained by discounting back the cash flow using the forward swap rate and credit spread for each grade.
- With one simulation of portfolio asset latent values, the CreditMetric model could estimate a change in mark-to-market portfolio value due to the credit migration change generated in the third step. In the case of a default event, an individual loss equals the product of exposure at default (EAD) and the loss given default (LGD).
- A distribution of such changes is obtained with more simulations. The corresponding portfolio economic capital is calculated by the difference between expected loss and the VaR at a certain confidence level.

Since we have no cash flow data available for the SME loan portfolio, we build a stylized portfolio in which each obligor is associated with a fictitious three-year coupon paying loan with face value \$100. Table 4.13 displays the characteristics of the artificial portfolio. Coupons are assigned according to Mählmann (2006) such that the price is not too far from par value. SME Internal ratings are mapped to external ones such that corresponding default probabilities approximately agree. The artificial portfolio is accordingly composed of obligors whose ratings are distributed across rating grades similar to that of the SME large loan portfolio in *Manufacturing* sector. Credit spreads and the one-year forward swap rate that prevailed on March 28, 2014 are taken. A 45% recovery rate and 0.21 factor loading are assumed.

We calculate the portfolio credit loss for each quarter's conditional migration matrix. The third quarter of 2008 is chosen to illustrate the exercise because the one-factor conditional migration matrix outperforms the other two benchmarks in terms of distance via risk-sensitive criteria but not mobility-based criteria. We would like to examine whether the one-factor conditional migration matrices using different criteria have the same performance in terms of credit risk. Since the credit loss analysis is commonly set horizon to one year, we convert the quarterly transition matrix of 2008Q3 to an annual transition matrix.

Monte Carlo simulations (100,000 trials) are used to obtain the portfolio value distributions. Figure 4.5 shows distributions based on the Cohort migration (left side) and the Duration migration (right side) respectively. Each side compares the distribution obtained from model forecasted matrix using different metrics (D1, D2 and SVD) with the one from empirical migration. We see that the distributions are irrespective of the metric chosen to calibrate the weight ω in the one-factor model. Compared to the distribution obtained from the migration forecasts of Naive I and II approaches, Figure 4.6 shows that the one obtained from one-factor model migration forecast via metric D1 is closer to the distribution obtained from the empirical migration matrix for either the Cohort or the Duration migrations. We can see this result again in the credit loss calculation that follows. It is worth noting that, in terms of distance, the SVD metric chooses the Naive II approach as the best forecast for this quarter whereas the risk-sensitive metrics (D1 and D2) both found that the one-factor model performs the best.

Table 4.14 illustrates the results of credit loss from credit quality migration, including the expected loss, VaR (Value at Risk) at 99% and 99.9% levels, and the corresponding economic capital (percentile loss from expected loss). The numbers in parentheses provide the ratio of results between the predicted migration matrices and the empirical ones. We find that credit loss of the artificial portfolio is underestimated (30-40% less for VaR, and 20-30% for economic capital) using migration matrices from the Naive approaches. However, the one-factor model gives closer results for VaR and economic capital. This example shows the importance of migration matrix quality for the accuracy of credit loss simulation.

4.5.2 Stressed Migration Matrix

Since the financial crisis (2007-2009), the importance of stress testing has magnified. Among the different categories of stress tests, macroeconomic stress tests play an important role. Not only stressed PD (probability of default), but also stressed migration matrices need to be calculated depending on the specific risk application. The one-factor model examined in this chapter provides an approach to stress a migration matrix using changes in macroeconomic variables. In the case of a historical stress test scenario, observed evolutions of macroeconomic indicators in the past are applied to simulate the impact on the credit portfolio, while in the case of a hypothetical stress test scenario experts determine a certain situation or evolution of the macroeconomic indices depending on the exact purpose of the test.

Following Miu and Ozdemir (2008), we apply the one-factor model to compute the stressed migration matrices conditional on a hypothetical stress scenario for 2008Q3 as an example. Specifically, the hypothetical scenario is defined as:

- A change in $GCAN5YR$ (Canadian 5 year bond yield) which is one standard deviation below the average; together with

- A change in $CAEXR$ (exchange rate) which is one standard deviation below the average; and
- A change in VIX (S&P volatility index) which is one standard deviation below the average; and
- A change in $MPROD$ (manufacturing production) which is one standard deviation above the average; and
- A change in $SLFSI$ (St. Louis stress index) which is one standard deviation above the average and
- A change in $SPLR\_BCF$ (spread of loan rates over cost for small firms) which is one standard deviation above the average.

The standard deviation and average figures used in the hypothetical scenario is based on historical data 1998Q1 to 2008Q2. The stressed migration matrices based on the criterion D1 for determining weight ω for the Cohort and the Duration estimates are reported in Table 4.15 - 4.16. As compared with the baseline migration matrix (without stress scenario), the stressed migration matrices have as expected increasing probabilities of downgrading and default as expected, around 1.5-2.5 times higher. For both the Cohort and Duration estimates, the increase in downgrades of the stressed migration matrices mainly come from a decrease in the upgrade part. The stressed diagonal probabilities are slightly reduced compared to those of the baseline matrices.

Using the same CreditMetric approach described above, we obtain the stressed economic capital under the hypothetical scenario. As shown in Table 4.17, the economic capital under the stressed scenario are double those of the baseline situation. With regard to the stressed economic capital between the two estimation methods, the Cohort one is higher. This result is in line with studies in Chapter 3 that the Cohort methodology leads to lower diagonal probabilities.

4.6 Conclusions

Credit migration matrices play an important role in portfolio credit value assessment and stress testing analysis. There have been several studies to estimate migration matrices conditional on macroeconomic variables to highlight rating changes over the business cycle. However, existing research is based on external rating data. In this chapter, we implement the conditional approach to forecast migration matrices based on SME internal rating data from a Canadian *Financing Company* from 1998Q1 to 2009Q4. More specifically, both the Cohort and the Duration migration matrices are used for the modeling. Model performance both in-sample and out-of-sample periods are examined using both mobility-based and risk-sensitive metrics. The conditional migration matrices are then compared in assessing portfolio credit values and used in a stress testing experiment.

Following Kim (1999) and Trück (2008), we use a one-factor structural model which as suggested by the Basel II requirement that is compatible with the internal rating system of banks so that both PDs and migration rates can be stressed within one framework. Another advantage of one-factor model is to simplify by model development if the financial institution already has a PD forecast model incorporating macroeconomic variables. Using *Manufacturing* sectoral data in large (over \$1 million) loan group in the model calibration, we calibrate a one-factor model in two steps: building a credit cycle index and conditioning the transition matrices. In the first step, we design a credit cycle index Z_t based on probability of default (PD) to represent the credit state of the financial market and forecast Z_t by macroeconomic variables in a linear regression model. Then in the second step, a credit-change indicator Y_t related to Zt is defined by the one-factor model that includes a weight ω. The probability that Y signals a rating change is calibrated to the average transition rate. Conditional on Z_t, regression parameters and the weight parameter, an estimated migration matrix is constructed.

The in-sample (1998Q1-2007Q4) results confirms the superior performance of the one-factor model as compared with two benchmarks. The out-of-sample (2008Q1-2009Q4) results show that the one-factor model forecast is superior to the benchmarks only if the regression forecast has high accuracy in the first stage of modeling. We have several findings specific to the SME internal rating data:

1. Within a choice set of 26 macroeconomic and financial series taken from four categories (Financial Markets, Stock Markets, National Accounts and Composite Indices) to model the obligors' default behavior, the macroeconomic variables that survived a selection process are similar using either the Cohort or the Duration migration matri-

ces. The five significant economic variables based on the Cohort migration estimates are: “Canadian 5 years government bond”, “Canadian dollar effective exchange rate index”, “SPX volatility index”, “Manufacturing production”, and “St. Louis financial stress index”. The four macroeconomic variables based on the Duration estimates include the same first three variables as those in the Cohort specification and “Spreads of Loan Rates over Banks’ Cost for Small Firms”. In contrast to the previous results using external rating data, these variables are related more to the financial market instead of standard macroeconomic variables such as unemployment rate and GDP.

2. The variables selected explain less default behavior for the SME loan portfolio than that associated with a large corporate bond portfolio. The adjusted R-square statistic of regression step in our study is only 0.65 and 0.44 using the Cohort and the Duration migration estimates respectively, lower than the result of 0.98 using Standard & Poor’s in Trück (2008). Besides the limited data time span, the diversification of SME loan portfolio might cause their default behavior less sensitive to the general economic conditions than that of large corporations.

3. The in-sample weights w (0.09-0.13 for the Cohort migration matrices and 0.13-0.16 for the Duration migration matrices) for the one-factor model based our SME internal rating data are lower 20-50% than the results (0.21-0.23) using external rating data (Trück, 2008). It shows that the credit migration behaviors of our SME loans could be explained less by the PD-based credit cycle index compared with those of large corporate bonds. Diversification of SME loan portfolio might contribute to such higher obligor-specific effect in the one-factor model. With data availability, one could reflect the heterogeneity of the sample portfolio by calibrating w for each rating category.

4. The out-of-sample quarterly performance based on three distance metrics of migration matrices (risk-sensitive metrics D1, D2 and mobility-based metric SVD) is provided for first time in this study. Based on the metrics D1 and D2, the one-factor model outperforms the two naive approaches for the quarters when the regression produce accurate forecasts. In contrast, we cannot identify the one-factor model as a superior method based on the SVD distance between the modeled and empirical migration matrices. However, taking 2008Q3 as an example, the predicted migration matrices from one-factor model provides the closest distribution of portfolio credit value change relative to the empirical one when the one-factor model is outperformed by the naive benchmarks according to the SVD metric. In terms of economic relevance, we doubt

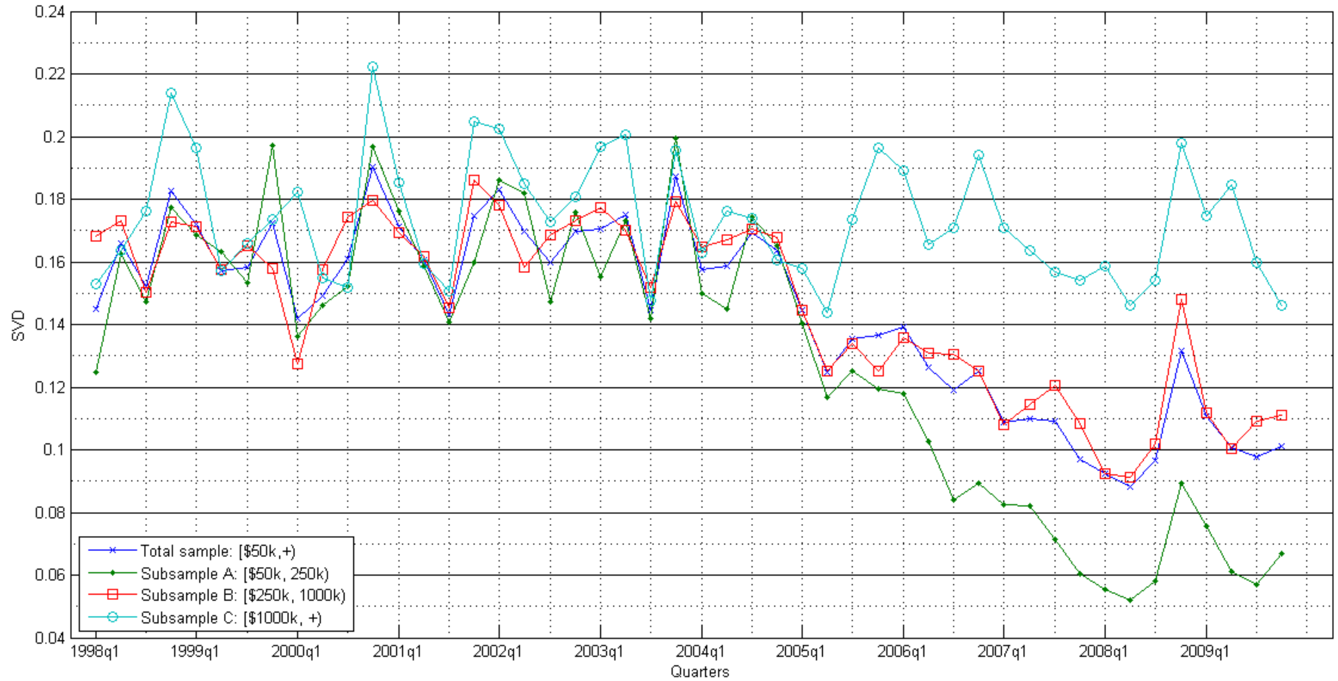
the capability of mobility-based metric SVD to measure the difference of migration matrices.

Finally, we apply the one-factor modeled migration matrices to evaluate a artificial portfolio which has the same rating distribution as the SME sample data used in this chapter. Using CreditMetric model, 100,000 Monto Carlo simulations are performed to generate a portfolio credit value distribution. Compared to the credit loss obtained from migration matrices of two naive approaches, we find that the one obtained from conditional migration matrices reduce the deviation by 4-5 times. The result shows the importance of migration matrix quality for the accuracy of credit loss simulation.

The one-factor model is also used to assess the credit loss change of a portfolio under a stressed scenario. The stress test is exercised under a hypothetical scenario in which one standard deviation above average value is taken for all relevant macroeconomic and financial variables. Using 2008Q3 as an example, we determined the stressed migration matrix and compare the resulting stressed economic capital for the Cohort and Duration estimates. Downgrade probabilities are increased in the stressed scenarios and economic capital almost doubles. In line with the results in Chapter 3, the Duration estimates provide lower economic capital than the Cohort estimates.

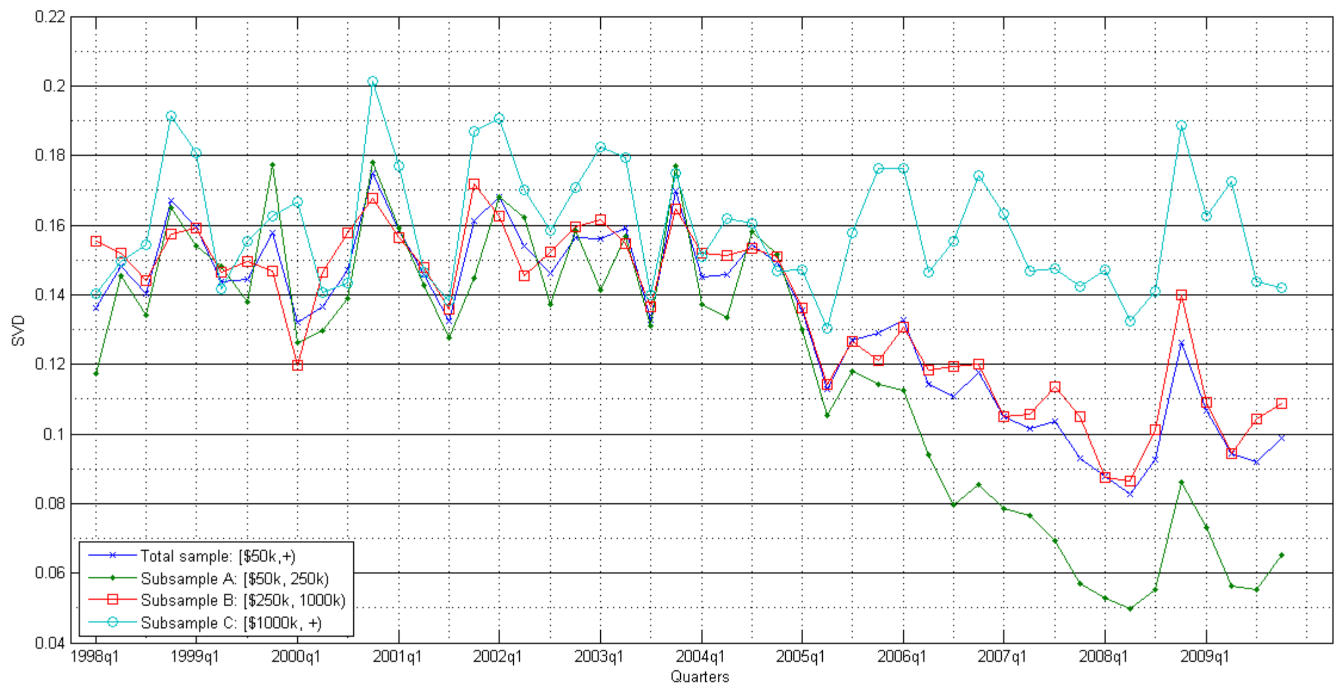
In all, we conclude that the one-factor model using the internal rating data of the SME loan portfolio is appropriate for forecasting credit migration matrices conditional on macroeconomic and financial variables. However, its ultimate success relies on the forecast accuracy of the PD-based credit cycle index.

Figure 4.1: Mobility Metric SVD of Quarterly Migration Matrices via the Cohort method by Sample Groups for Manufacturing Sector



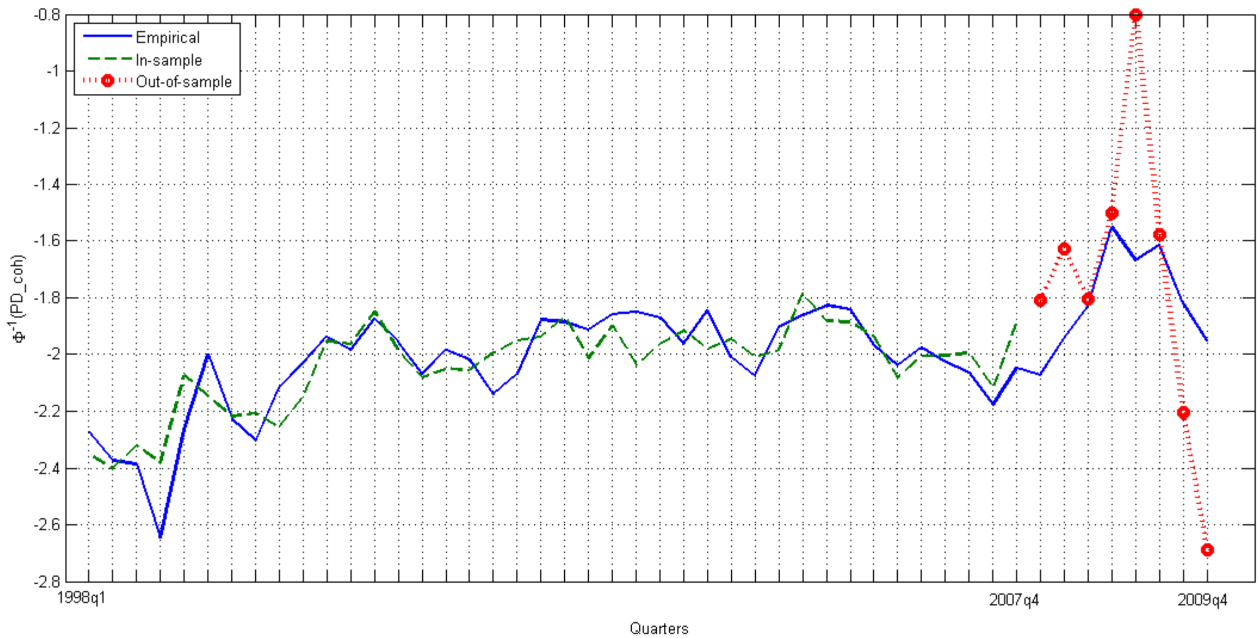
Using the metric of SVD (Single Value Decomposition), the figure demonstrates the implied mobility of quarterly migration matrices via the Cohort method for different sample groups from 1998 to 2009. With the rating reivew policy change for loans under \$1000k in 2005, we can see a sudden drop of migration mobility for sample group A, B after 2005.

Figure 4.2: Mobility Metric SVD of Quarterly Migration Matrices via the Duration method by Sample Groups for Manufacturing Sector



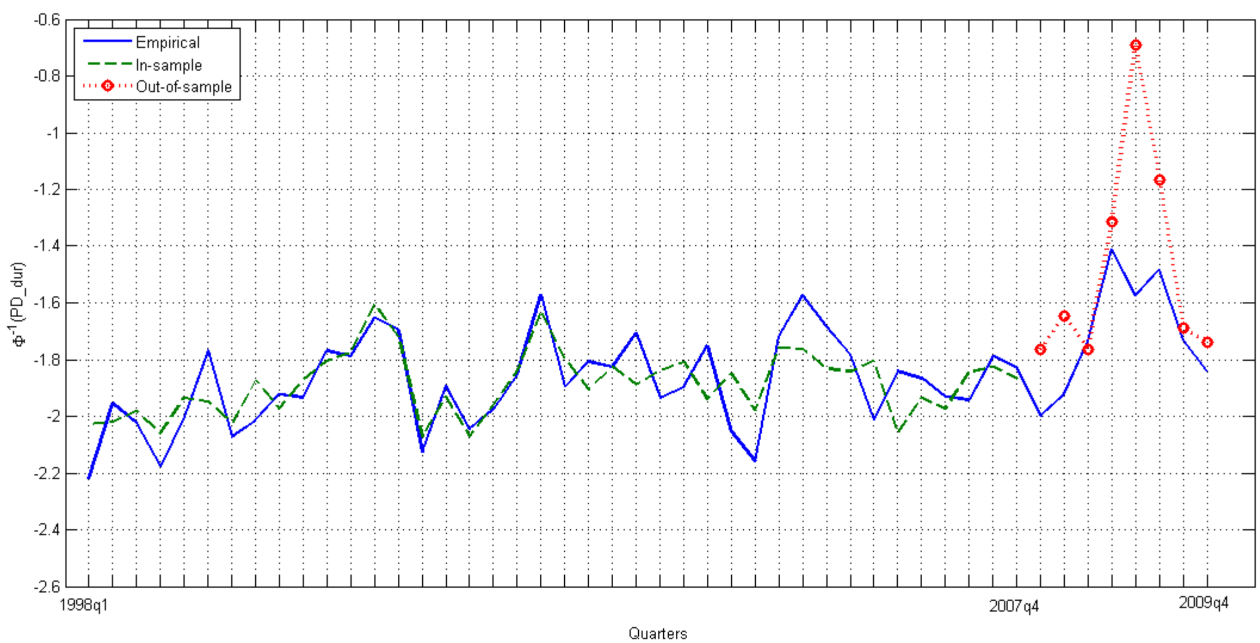
Using the metric of SVD (Single Value Decomposition), the figure demonstrates the implied mobility of quarterly migration matrices via the Duration method for different sample groups from 1998 to 2009. With the rating reivew policy change for loans under \$1000k in 2005, we can see a sudden drop of migration mobility for sample group A, B after 2005.

Figure 4.3: In-sample and Out-of-sample Regression Plots with the Cohort Migration Estimates



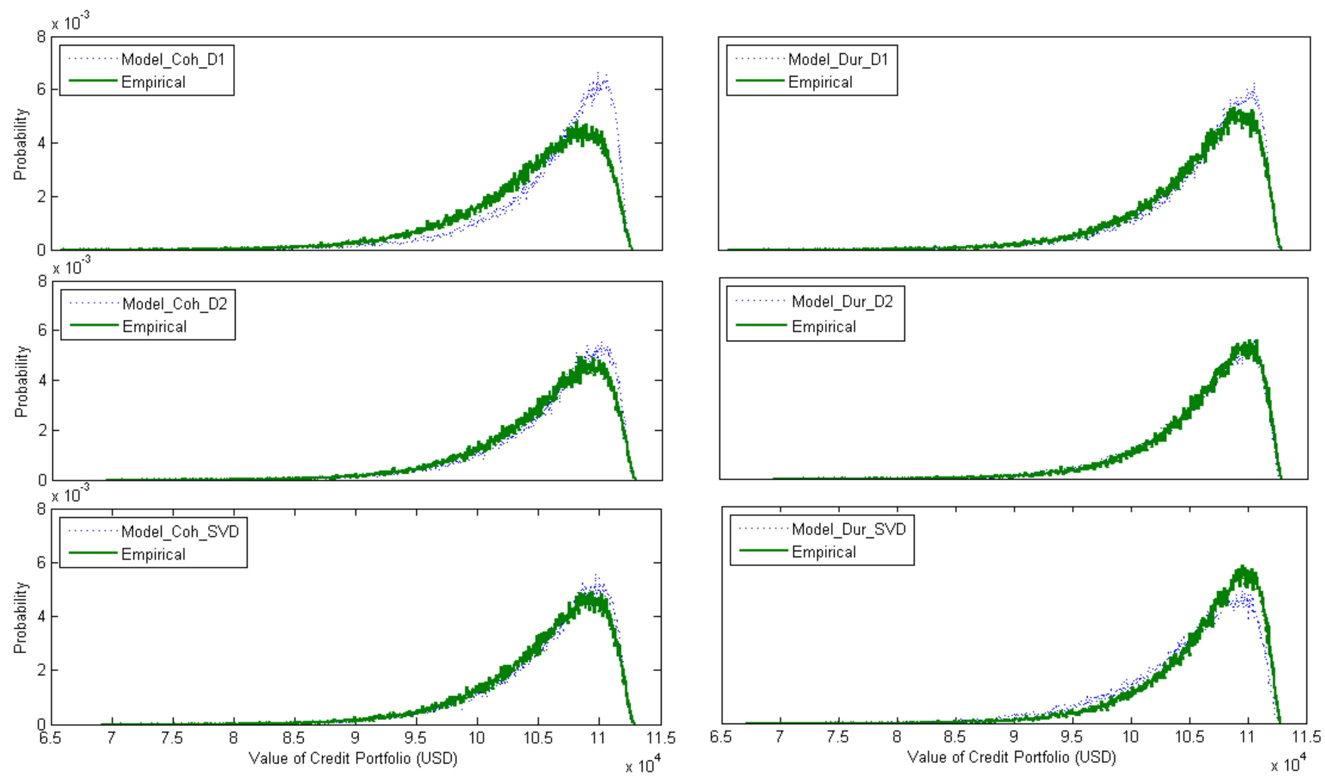
The figure shows regression plot of $\Phi^{-1}(PD_t)$ using the Cohort migration estimates from 1998Q1 to 2009Q4. Solid line represent empirical value; the dash line represents the fitted value for the in-sample period (1998Q1-2007Q4) and dot line with circle mark is predicted value for out-of-sample period (2008Q1-2009Q4).

Figure 4.4: In-sample and Out-of-sample Regression Plots with the Duration Migration Estimates



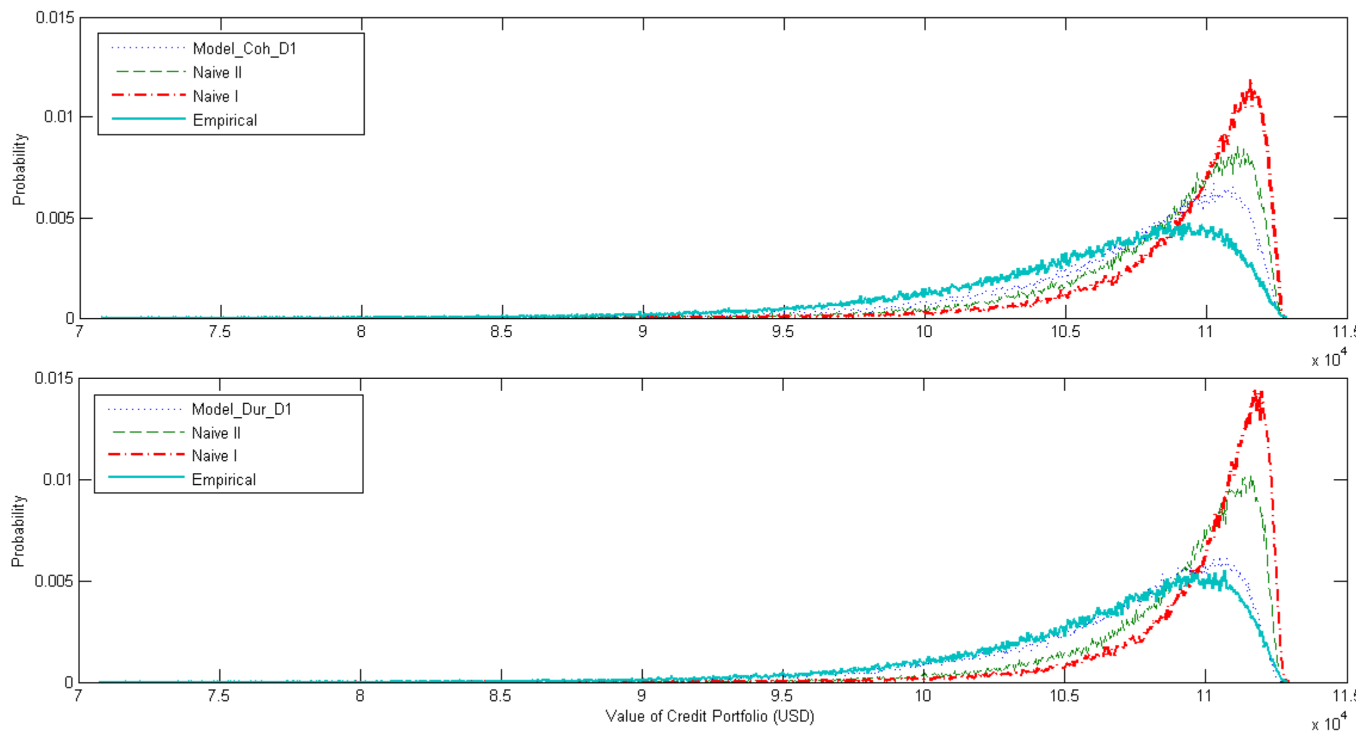
The figure shows regression plot of $\Phi^{-1}(PD_t)$ using the Duration migration estimates from 1998Q1 to 2009Q4. Solid line represent empirical value; the dash line represents the fitted value for the in-sample period (1998Q1-2007Q4) and dot line with circle mark is predicted value for out-of-sample period (2008Q1-2009Q4).

Figure 4.5: Simulated Distributions of Portfolio Credit Value Changes Obtained from the One-factor Model Forecast of Migration and Empirical Migration



The figure shows distributions of portfolio credit value changes through empirical and one-factor model forecasted migration matrices for 2008Q3. The model forecasts are via three metrics (D1, D2, SVD). Left side is based on the Cohort migration and right side is based on the Duration migration. Monte Carlo simulations (100,000 trials) are used to obtain the portfolio value distributions one year hence. The initial exposure of artificial portfolio is $104,100.

Figure 4.6: Simulated Distribution of Portfolio Credit Value Changes Obtained from Migration Forecasts of Different Approaches



The figure shows distributions of portfolio credit value changes through empirical, one-factor model (via Metric D1), and two Naive approaches migration matrices for 2008Q3. Left side is based on the Cohort migration and right side is based on the Duration migration. Monte Carlo simulations (100,000 trials) are used to obtain the portfolio value distributions one year hence. The initial exposure of artificial portfolio is $104,100.

Table 4.1: Description and Resources of Macroeconomic Variables

Category	Variables	Description	Ticker in Bloomberg/Resource
	GCAN3M	Canadian 3 month government bond yield	GCAN3M Index
	GCAN10YR	Canadian 10 year government bond yield	GCAN10YR Index
	GCAN5YR	Canadian 5 year government bond yield	GCAN5YR Index
	BINTR	Bank of Canada administered interest rate	Statistics Canada: STAC_v122530
	SP10_3	Spread between Canadian government 10-year bond and 3-month bond	GCAN10YR-GCAN3M
	SP5_3	Spread between Canadian government 5-year bond and 3-month bond	GCAN5YR-GCAN3M
	SP3B_3A	Spread between Moody's Seasoned Aaa/Baa US corporate bond yield	Federal Reserve Bank of St. Louis
Financial Markets	SP3B_10YR	Spread between Moody's Seasoned Baa US corporate bond yield and 10-Year treasury yield	Federal Reserve Bank of St. Louis
	SPLR_BCF	Spreads of U.S. loans rates over banks' cost of funds Small Firms	Federal Reserve Bank of St. Louis
	CANEXR	Canadian dollar effective exchange rate index	CERIINDX Index
	RUSSELL	Return of Russell2000 Index	Yahoo historical download
	SPTSX	Return of S&P/TSX COMPOSITE IDX	SPTSX Index
Stock Markets	VIX	CBOE VOLATILITY INDX Conveyed by S&P 500 Index (SPX) option prices	VIX Index
	SPPE	S&P 500 Price /Earnings ratio	Yale Department of Economics
	RGDP	Canada real GDP (YoY%)	EHGDCA Index
	UNEMR	Canada unemployment rate (%)	EHUPCA Index
	CCI	OECD Canada leading Ind. consumer confidence Norm SA	OECAA024 Index
	CPI	Canada consumer price index (YoY %)	EHPICA Index
	IPRODPR	STCA canda industrial product price MoM NSA	CAIPMOM Index
	IPROD	Industry production	Statistics Canada: STAC_v41881184
	CAPUTL	STCA Canada industrial capacity utilization	CACAPUTL Index
National Accounts	MPROD	OECD production in total manufacturing for Canada	CANPROMANMISMEI
	MPPI	OECD Producer prices index: Total Manufacturing for Canada	PIEAMP01CAA661N
	MPMI	Manufacturing PMI (Purchasing Manager Index)	NAPM
Composite Indices	CFNAI	Chicago Fed National Activity Index	CFNAI Index
	STLFSI	St. Louis Financial Stress Index	Federal Reserve Bank of St. Louis

The table provide description and resources of 26 macroeconomic variables used in modeling the conditional migration matrices.

Table 4.2: **Statistics of Macroeconomic Variables**

Variable	Mean	StdDev	Min	Max
Financial Markets				
GCAN3M	0.05	0.89	-1.63	1.67
GCAN5YR	0.01	0.53	-1.23	1.08
GCAN10YR	-0.01	0.36	-0.80	0.73
BINTR	0.04	0.93	-1.64	1.71
SP5_3	-0.05	0.72	-1.56	1.37
SP10_3	-0.06	0.86	-1.54	1.63
SP3B_3A	0.00	0.41	-1.44	0.88
SP3B_10YR	0.00	0.22	-0.35	0.61
SPLR_BCF	0.83	24.35	-48.10	56.99
CAEXR	0.01	22.47	-26.70	90.64
Stock Markets				
RUSSELL	-0.29	75.32	-195.61	152.25
SPTSX	0.00	134.46	-327.81	273.53
VIX	0.09	6.56	-8.70	17.79
SPPE	0.26	3.65	-7.27	7.32
National Accounts				
RGDP	0.02	1.17	-2.74	1.68
UNEMR	-0.02	0.54	-0.82	1.22
CCI	-0.04	1.07	-3.17	2.06
CPI	-0.01	0.79	-1.93	2.02
IPRODPR	0.00	0.88	-2.29	1.73
IPROD	0.00	6.45	-16.83	12.70
CAPUTL	0.02	1.70	-5.43	2.77
MPROD	0.10	3.94	-8.74	9.58
MPPI	0.00	0.02	-0.04	0.06
MPMI	-0.11	4.62	-11.75	9.29
Composite Indices				
CFNAI	-0.41	1.01	-3.77	0.91
SLFSI	0.19	1.09	-0.98	4.40

The table provide statistics of the filtered macroeconomic covariates used in modeling the conditional migration matrices. See detail discussion in Section 4.4.1.1.

Table 4.3: Correlations Among Macroeconomic Variables

	GCAN3M	GCAN5YR	GCAN10YR	BINTR	SP5_3	SP10_3	SP3B_3A	SP3B_10YR	SPLR_BCF	CAEXR	RUSSELL	SPTSX	VIX	SPPE	RGDP	UNEMR	CCI	CPI	IPRODPR	IPROD	CAPUTL	MPROD	MPPI	MPMI	CFNAI	SLFSI
GCAN3M	1																									
GCAN5YR	0.63	1																								
GCAN10YR	0.33	**0.92**	1																							
BINTR	**0.97**	0.64	0.35	1																						
SP5_3	-0.6	0.02	0.33	-0.6	1																					
SP10_3	-0.7	-0.2	0.13	-0.7	**0.97**	1																				
SP3B_3A	0.35	0.52	0.44	0.33	-0.1	-0.2	1																			
SP3B_10YR	-0.5	-0.6	-0.4	-0.5	0.26	0.41	-0.8	1																		
SPLR_BCF	-0.1	-0.2	-0.1	-0.1	0.23	0.27	-0.6	0.73	1																	
CAEXR	-0.3	-0	0.08	-0.3	0.32	0.33	0.02	0.26	0.22	1																
RUSSELL	0.63	0.64	0.42	0.65	-0.3	-0.5	0.66	-0.8	-0.4	-0.3	1															
SPTSX	0.69	0.78	0.57	0.76	-0.3	-0.5	0.56	-0.7	-0.2	-0.2	0.84	1														
VIX	-0.2	-0.5	-0.4	-0.2	0.07	0.2	-0.7	0.68	0.67	0.12	-0.7	-0.6	1													
SPPE	0.55	0.59	0.38	0.55	-0.3	-0.5	0.57	-0.6	-0.4	-0	0.75	0.72	-0.7	1												
RGDP	0.53	0.51	0.33	0.6	-0.4	-0.5	0.29	-0.4	-0.2	-0.2	0.4	0.58	-0.3	0.55	1											
UNEMR	-0.7	-0.5	-0.3	-0.8	0.33	0.47	0.01	0.23	-0.3	0.36	-0.4	-0.6	-0	-0.4	-0.6	1										
CCI	0.3	0.53	0.5	0.32	-0.1	-0.2	0.69	-0.6	-0.5	-0.1	0.55	0.5	-0.6	0.46	0.42	0.01	1									
CPI	0.35	0.26	0.22	0.42	-0	-0.1	-0.2	0.14	0.36	-0.1	-0.1	0.15	0.28	-0.1	0.39	-0.6	-0.3	1								
IPRODPR	0.05	0.12	0.07	0.06	-0.1	-0.1	0.25	-0.1	-0.1	-0	0.25	0.28	-0.1	0.23	0.16	-0.1	0.27	-0.1	1							
IPROD	0.64	0.66	0.47	0.72	-0.3	-0.5	0.28	-0.5	-0.1	-0.3	0.63	0.81	-0.3	0.61	0.77	**-0.8**	0.41	0.34	0.24	1						
CAPUTL	0.64	0.55	0.37	0.71	-0.3	-0.4	0.21	-0.5	-0.1	-0.4	0.52	0.68	-0.2	0.43	0.78	**-0.8**	0.36	0.46	0.09	**0.85**	1					
MPROD	0.7	0.61	0.4	0.75	-0.3	-0.5	0.17	-0.4	0.03	-0.3	0.52	0.75	-0.2	0.51	0.69	**-0.8**	0.15	0.48	0.09	**0.87**	**0.83**	1				
MPPI	0.28	0.3	0.26	0.33	-0	-0.1	-0.3	0.13	0.4	-0.1	-0	0.23	0.27	-0.1	0.2	-0.6	-0.4	0.6	-0.2	0.37	0.39	0.52	1			
MPMI	0.07	0.39	0.4	0.07	0.04	-0	0.65	-0.6	-0.6	0	0.35	0.38	-0.5	0.47	0.5	0.03	0.64	-0.2	0.23	0.39	0.35	0.17	-0.3	1		
CFNAI	0.28	0.32	0.21	0.27	-0.3	-0.4	0.65	-0.6	-0.6	-0.4	0.56	0.41	-0.6	0.47	0.34	0.02	0.66	-0.3	0.27	0.41	0.26	0.18	-0.4	0.58	1	
SLFSI	-0.4	-0.4	-0.3	-0.4	0.29	0.39	-0.7	0.81	0.73	0.51	-0.7	-0.5	0.72	-0.5	-0.3	0.09	-0.6	0.19	-0.2	-0.4	-0.3	-0.3	0.21	-0.5	-0.9	1

The table demonstrates correlations among 26 filtered macroeconomic variables. See Table 4.1-4.2 for the definition and statistics of variables.

Table 4.4: **Backward Selection of Macroeconomic Variables**

Cohort Migration						
Source	**SS**	**DF**	**MS**		**Number of obs**	40
					F(5, 34)	16.05
Model	0.882	5	0.176		**Prob > F**	0
Residual	0.374	34	0.011		**R-squared**	0.7024
					Adj R-squared	0.6586
Total	1.256	39	0.032		**Root MSE**	0.1049
	Coef	**Std Err**	**Std Coef**	**P>|t|**	**[95% Conf Interval]**	
$MPROD$	0.031	0.006	0.635	0.000	0.019	0.044
$SLFSI$	0.346	0.055	1.020	0.000	0.235	0.457
$GCAN5YR$	-0.128	0.047	-0.360	0.011	-0.225	-0.032
$CAEXR$	-0.012	0.002	-0.735	0.000	-0.015	-0.008
VIX	-0.018	0.005	-0.553	0.001	-0.027	-0.008
CONSTANT	-2.070	0.020		0.000	-2.111	-2.030

Duration Migration						
Source	**SS**	**DF**	**MS**		**Number of obs**	40
					F(4, 35)	8.97
Model	0.500	4	0.125		**Prob > FF**	0
Residual	0.488	35	0.014		**R-squared**	0.506
					Adj R-squared	0.450
Total	0.988	39	0.025		**Root MSE**	0.118
	Coef	**Std Err**	**Std Coef**	**P>|t|**	**[95% Conf**	**Interval]**
$MPROD$	0.037	0.007	0.813	0.000	0.022	0.051
$SLFSI$	0.300	0.075	1.834	0.000	0.148	0.453
$GCAN5YR$	-0.124	0.052	-0.368	0.022	-0.230	-0.019
$SPLR\_BCF$	-0.007	0.002	-0.934	0.001	-0.011	-0.003
$CONSTANT$	-1.863	0.020		0.000	-1.905	-1.822

The table describes the backward selection of macroeconomic variables for inverse CDF of default probability ($\Phi^{-1}(S_t)$) using the Cohort and the Duration estimates of the average PD respectively. Sample period covers from 1998Q1 to 2007Q4, 40 quarters. Independent variables are lagged one quarter. Standard coefficients are shown to better gauge the relative importance of selected variables. SS denotes sum of squares and DF denotes degree of freedom. MS represents mean squares, which is the sum of squares divided by their respective DF. Root MSE is the square root of mean square residual.

Table 4.5: Ratios between the Predicted and the Empirical Probabilities of Default

	PD_Coh	PD_Dur
2008q1	1.86	1.64
2008q2	1.99	1.73
2008q3	**1.06**	**0.92**
2008q4	**1.10**	**1.21**
2009q1	4.45	3.21
2009q2	**1.08**	1.74
2009q3	0.40	**1.09**
2009q4	0.14	1.28

The table exhibits the out-of-sample regression performance via the ratio of predicted PD and empirical PD. The ratios are calculated under two different methods of PD estimates, namely the Cohort method and the Duration method. The relative accuracy of predicted values are identified in bold.

Table 4.6: Historical Average Transition Rates and Associated Bins: Cohort

Rating	1.5	2	2.5	3	3.5	4	4.5	D
				Historical Average Transition Rates				
1.5	87.92%	3.40%	2.77%	2.00%	1.15%	1.36%	0.71%	0.69%
2	6.17%	80.77%	3.96%	3.97%	1.64%	1.66%	0.94%	0.90%
2.5	4.44%	4.06%	80.45%	4.57%	2.12%	2.04%	1.08%	1.24%
3	2.73%	3.22%	4.13%	80.51%	3.10%	2.94%	1.70%	1.68%
3.5	1.45%	2.02%	3.09%	5.62%	78.98%	4.46%	2.36%	2.02%
4	1.02%	0.95%	1.88%	3.97%	3.68%	81.75%	4.13%	2.61%
4.5	0.62%	0.68%	1.17%	2.20%	2.76%	4.91%	84.40%	3.26%
				Bins Corresponding to Historical Average Transition Rates				
1.5	(∞ , -1.17)	[-1.17 , -1.36)	[-1.36 , -1.56)	[-1.56 , -1.76)	[-1.76 , -1.91)	[-1.91 , -2.19)	[-2.19 , -2.46)	[-2.46 ,$-\infty$)
2	(∞, 1.54)	[1.54 , -1.12)	[-1.12 , -1.33)	[-1.33 , -1.63)	[-1.63 , -1.81)	[-1.81 , -2.08)	[-2.08 , -2.36)	[-2.36 ,$-\infty$)
2.5	(∞ , 1.70)	[1.70 , 1.37)	[1.37 , -1.22)	[-1.22 , -1.51)	[-1.51 , -1.71)	[-1.71 , -1.99)	[-1.99 , -2.24)	[-2.24 ,$-\infty$)
3	(∞, 1.92)	[1.92 , 1.55)	[1.55 , 1.27)	[1.27 , -1.31)	[-1.31 , -1.52)	[-1.52 , -1.82)	[-1.82 , -2.12)	[-2.12 ,$-\infty$)
3.5	(∞, 2.18)	[2.18 , 1.81)	[1.81 , 1.50)	[1.50 , 1.16)	[1.16 , -1.35)	[-1.35 , -1.70)	[-1.70 , -2.04)	[-2.04 ,$-\infty$)
4	(∞, 2.31)	[2.31 , 2.05)	[2.05 , 1.76)	[1.76 , 1.41)	[1.41 , 1.20)	[1.20 , -1.49)	[-1.49 , -1.94)	[-1.94 , $-\infty$)
4.5	(∞ , 2.50)	[2.50 , 2.22)	[2.22 , 1.96)	[1.96 , 1.67)	[1.67 , 1.44)	[1.44 , 1.15)	[1.15 , -1.84)	[-1.84 ,$-\infty$)

The table exhibits the thresholds and bins derived from the average quarterly Cohort migration matrices for the in-sample period from 1998Q1 to 2007Q4.

Table 4.7: Historical Average Transition Rates and Associated Bins: Duration

Rating	1.5	2	2.5	3	3.5	4	4.5	D
				Historical Average Transition Rates				
1.5	89.19%	3.16%	2.23%	1.87%	1.04%	1.24%	0.64%	0.65%
2	6.29%	82.33%	3.33%	3.31%	1.53%	1.36%	0.87%	0.99%
2.5	4.45%	3.89%	81.08%	3.89%	2.16%	1.98%	1.15%	1.39%
3	3.01%	3.25%	3.59%	81.60%	2.71%	2.69%	1.36%	1.79%
3.5	1.70%	2.10%	3.08%	4.94%	79.66%	4.00%	2.10%	2.42%
4	1.26%	1.21%	1.86%	3.96%	3.48%	81.83%	3.47%	2.92%
4.5	0.78%	1.04%	1.16%	2.21%	3.08%	3.88%	83.14%	4.72%
				Bins Corresponding to Historical Average Transition Rates				
1.5	(∞,-1.24)	[-1.24, -1.43)	[-1.43, -1.60)	[-1.60,-1.80)	[-1.80,-1.96)	[-1.96, -2.23)	[-2.23, -2.49)	[-2.49,$-\infty$)
2	(∞, 1.53)	[1.53,-1.21)	[-1.21, -1.40)	[-1.40, -1.67)	[-1.67, -1.85)	[-1.85, -2.08)	[-2.08, -2.33)	[-2.33,$-\infty$)
2.5	(∞,1.70)	[1.70, 1.38)	[1.38, -1.25)	[-1.25, -1.50)	[-1.50, -1.69)	[-1.69, -1.95)	[-1.95, -2.20)	[-2.20,$-\infty$)
3	(∞, 1.88)	[1.88, 1.53)	[1.53, 1.29)	[1.29, -1.37)	[-1.37, -1.57)	[-1.57, -1.86)	[-1.86, -2.10)	[-2.10,$-\infty$)
3.5	(∞, 2.12)	[2.12, 1.77)	[1.77, 1.49)	[1.49, 1.18)	[1.18, -1.37)	[-1.37, -1.69)	[-1.69, -1.97)	[-1.97,$-\infty$)
4	(∞,2.24)	[2.24, 1.96)	[1.96, 1.71)	[1.71, 1.39)	[1.39, 1.19)	[1.19, -1.52)	[-1.52, -1.89)	[-1.89,$-\infty$)
4.5	(∞,2.42)	[2.42, 2.09)	[2.09, 1.88)	[1.88, 1.63)	[1.63, 1.39)	[1.39, 1.17)	[1.17, -1.67)	[-1.67,$-\infty$)

The table exhibits the thresholds and bins derived from the average quarterly Duration migration matrices for the in-sample period from 1998Q1 to 2007Q4.

Table 4.8: In-sample Estimated Weights ω for the Credit Cycle Index Z by Different Metrics and Migration Matrices Estimates

Optimization Criteria	D_1	D_2	SVD
Weights_Coh	0.086	0.125	0.130
Weights_Dur	0.127	0.158	0.167

The table provides the in-sample weights estimation for the credit cycle index Z_t by mobility-based metric (SVD) and risk-sensitive metrics ($D1$ and $D2$). The weights represent the influence of Z_t on the credit-change indicator Y based on the Cohort estimates and the Duration estimates of migration matrices. The in-sample period covers from 1998Q1 to 2007Q4.

Table 4.9: In-sample Results for Mean Absolute Errors (MAE) by Different Metrics and Migration Matrices Estimates

		Cohort			Duration		
Metrics		Model	Naive I	Naive II	Model	Naive I	Naive II
D1	MAE	**0.744**	1.113	1.046	**0.507**	1.066	0.984
	Stdev	(0.59)	(0.65)	(0.95)	(0.38)	(0.58)	(0.83)
	%(Model/Naive)		67%	71%		48%	52%
D2	MAE	**4.553**	7.779	7.450	**2.891**	7.473	7.102
	Stdev	(3.34)	(5.35)	(6.76)	(2.35)	(4.76)	(5.90)
	%(Model/Naive)		59%	61%		39%	41%
SVD	MAE	**0.015**	0.017	0.020	**0.014**	0.015	0.017
	Stdev	(0.01)	(0.01)	(0.02)	(0.01)	(0.01)	(0.01)
	%(Model/Naive)		87%	76%		99%	85%

The table exhibits in-sample (1998Q1-2007Q4) results of mean absolute errors and related standard deviation for migration matrices forecasts according to different metrics and estimation approaches. The left panel demonstrates the results based on the Cohort migration matrices and right panel shows the results based on the Duration matrices. To predict migration matrix, Naive I is the average migration matrix over the in-sample period and Naive II is the previous period's migration matrix. The smallest MAE figures are highlighted in bold. Note that the mean error or standard deviation of the errors for different indices within the columns cannot be compared because of a different scale. However, the results in the rows can be compared and provide the forecasting performance in comparison to other approaches.

Table 4.10: Out-of-sample Re-estimated Weights of Credit Cycle Index based on different Metrics and Migration Matrices Estimates

	Cohort			Duration		
Metrics	D1	D2	SVD	D1	D2	SVD
2008Q1	0.086	0.125	0.130	0.127	0.158	0.167
2008Q2	0.086	0.124	0.129	0.117	0.156	0.178
2008Q3	0.082	0.119	0.128	0.122	0.151	0.188
2008Q4	0.083	0.125	0.130	0.120	0.159	0.133
2009Q1	0.101	0.167	0.136	0.144	0.212	0.186
2009Q2	0.094	0.150	0.138	0.152	0.215	0.189
2009Q3	0.113	0.180	0.141	0.157	0.219	0.179
2009Q4	0.114	0.180	0.141	0.169	0.220	0.182

The table provides the out-of-sample weights estimation for the credit cycle index Z_t. The weights represent the influence of Z_t on the credit-change indicator Y based on different risk-sensitive metrics. The out-of-sample period covers from 2008Q1 to 2009Q4.

Table 4.11: Out-of-sample Results of Absolute Error Based on Different Metrics: Cohort

	D1			D2			SVD		
	Model	Naive I	Naive II	Model	Naive I	Naive II	Model	Naive I	Naive II
2008Q1	1.74	**0.34**	0.58	14.64	**3.00**	7.42	0.017	0.013	**0.004**
2008Q2	1.54	**1.18**	1.51	16.82	**7.80**	10.67	0.045	0.025	**0.013**
2008Q3	**0.88**	2.25	1.11	**7.19**	18.67	11.18	0.021	0.016	**0.008**
2008Q4	**2.24**	5.84	3.64	**13.80**	50.18	31.97	**0.006**	0.029	0.044
2009Q1	**0.98**	2.21	3.47	**5.30**	16.62	32.16	**0.003**	0.004	0.023
2009Q2	**1.34**	4.40	2.24	**4.05**	36.36	20.18	**0.008**	0.014	0.010
2009Q3	3.39	**1.94**	2.35	25.14	**12.00**	23.40	0.011	**0.011**	0.025
2009Q4	3.99	**0.21**	2.11	25.72	**2.14**	13.91	0.074	0.024	**0.014**

The table provides the performances of migration matrices forecast for the out-of-sample period from 2008Q1 to 2009Q4 by different forecasting approaches and metrics. The data is based on the Cohort migration estimates. To predict migration matrix, Naive I is the average migration matrix over all previous periods and Naive II is the previous period's migration matrix. Best results for each quarter are in bold.

Table 4.12: Out-of-sample Results of Absolute Error Based on Different Metrics: Duration

	D1			D2			SVD		
	Model	Naive I	Naive II	Model	Naive I	Naive II	Model	Naive I	Naive II
2008Q1	2.16	**0.25**	0.32	17.71	**2.20**	4.62	0.010	0.010	**0.005**
2008Q2	1.76	**0.86**	1.11	18.15	**5.84**	7.98	0.025	0.024	**0.015**
2008Q3	**0.00**	1.85	1.02	**0.40**	15.38	9.77	0.016	0.016	**0.008**
2008Q4	**0.29**	5.24	3.44	**10.65**	44.58	29.65	**0.027**	0.033	0.048
2009Q1	**1.77**	2.23	2.85	**13.57**	17.04	26.16	0.032	**0.006**	0.026
2009Q2	5.09	3.66	**1.49**	71.72	30.00	**13.40**	0.017	0.016	**0.010**
2009Q3	**1.07**	1.83	2.02	**3.56**	11.07	20.62	0.015	**0.013**	0.029
2009Q4	2.91	**0.33**	2.11	25.83	**3.06**	14.49	0.014	0.014	**0.002**

The table provides the performances of migration matrices forecast for the out-of-sample period from 2008Q1 to 2009Q4 by different forecasting approaches and metrics. The data is based on the Duration migration estimates. To predict migration matrix, Naive I is the average migration matrix and Naive II is the previous migration matrix. Best results for each quarter are in bold.

Table 4.13: **Characteristics of Sample Portfolios and Calibration of CreditMetric Model**

Internal rating	S&P scale	Coupon(annual)	Distribution
1.5	B+	5.5	308
2	B+	6	174
2.5	B+	6.5	138
3	B	7	133
3.5	B	7.5	111
4	B	8	92
4.5	B-	8.5	85
Total			1041

The table exhibits the characteristics of constructed portfolio consisting of fictitious three-year, $100, coupon paying loans (one per obligor) and the year-end values are computed using conditional, unconditional and empirical migration matrices. Internal ratings are mapped to external ones as shown in the second column. Coupons are assigned referring to Mählmann (2006) such that pricing is close to par value. The distribution of obligors across rating grades for 2008Q3 is shown in the last column.

Table 4.14: **Simulated Portfolio Credit Loss ($)**

	Empirical	Naive I	Naive II	Model		
				D1	D2	SVD
Cohort Migrations						
EL	6,500	2,592	3,642	4,810	5,868	6,131
		(0.40)	(0.56)	(0.74)	(0.90)	(0.94)
VAR (99%)	23,430	14,189	16,751	20,331	23,131	23,779
		(0.61)	(0.71)	(0.87)	(0.99)	(1.01)
VAR (99.9%)	30,740	21,393	23,858	27,956	30,989	31,651
		(0.70)	(0.78)	(0.91)	1.01)	(1.03)
EC (99%)	16,929	11,597	13,108	15,521	17,263	17,648
		(0.68)	(0.77)	(0.92)	(1.02)	(1.04)
EC (99.9%)	24,239	18,801	20,216	23,147	25,120	25,520
		(0.78)	(0.83)	(0.95)	(1.04)	(1.05)
Duration Migrations						
EL	5,697	2,043	3,009	5,191	6,020	7,126
		(0.36)	(0.53)	(0.91)	(1.06)	(1.25)
VAR (99%)	21,740	12,582	15,141	21,263	23,331	25,987
		(0.58)	(0.70)	(0.9)	(1.07)	(1.20)
VAR (99.9%)	29,139	19,668	22,114	28,975	31,186	33,789
		(0.67)	(0.76)	(0.99)	(1.07)	(1.16)
EC (99%)	16,043	10,539	12,132	16,072	17,311	18,860
		(0.66)	(0.76)	(1.00)	(1.08)	(1.18)
EC (99.9%)	23,443	17,625	19,105	23,784	25,165	26,662
		(0.75)	(0.81)	(1.01)	(1.07)	(1.14)

Credit risk capital for a one-year horizon computed by a one-factor version of CreditMetrics. The upper panel results are based on the Cohort migration matrices and the bottom panel results are based on the Duration migration matrices. The sample portfolios is described in Table 4.13. Monte Carlo simulations (100,000 trials) are used to obtain the portfolio value distributions one year hence. Value at Risk (VaR) and Economic Capital (EC) are calculated at 99% and 99.9% percentile. Expected Loss (EL) is also illustrated at the bottom row. The numbers in parentheses provide the ratio of results between the predicted migration matrices and the empirical ones.

Table 4.15: Stressed Migration Matrix: Cohort

Baseline Migration Matrix

Ratings	1.5	2	2.5	3	3.5	4	4.5	Default
1.5	**86.0**	4.0	2.9	2.4	1.3	1.8	0.8	0.8
2	5.7	**80.0**	4.1	4.4	1.7	1.8	1.1	1.2
2.5	3.8	3.6	**79.0**	4.9	2.7	2.5	1.4	2.0
3	2.4	3.0	3.6	**80.1**	3.4	3.5	1.7	2.2
3.5	1.2	1.8	2.9	4.9	**78.0**	5.5	2.8	3.0
4	0.9	1.0	1.7	3.8	3.6	**81.0**	4.7	3.4
4.5	0.4	0.8	1.0	2.2	3.2	3.8	**83.2**	5.3

Stressed Migration Matrix

Ratings	1.5	2	2.5	3	3.5	4	4.5	Default
1.5	**76.8**	5.6	4.5	3.9	2.3	3.3	1.6	2.0
2	2.7	**73.7**	5.8	6.7	2.9	3.2	2.3	2.8
2.5	1.7	2.0	**73.8**	7.0	4.2	4.3	2.7	4.4
3	1.0	1.5	2.0	**76.6**	5.2	5.8	3.0	4.8
3.5	0.5	0.8	1.5	2.8	**75.0**	8.3	4.8	6.3
4	0.3	0.4	0.8	2.1	2.1	**79.6**	7.7	7.0
4.5	0.1	0.3	0.5	1.1	1.7	2.3	**83.7**	10.3

Using the Cohort migration matrices, the table reports the conditional transition migration matrices under the stress events described in section 4.5.2. The baseline migration matrix without stress scenario is also presented for comparison.

Table 4.16: Stressed Migration Matrix: Duration

Baseline Migration Matrix								
Ratings	1.5	2	2.5	3	3.5	4	4.5	Default
1.5	**88.6**	3.3	2.3	1.9	1.1	1.3	0.7	0.7
2	5.9	**82.1**	3.4	3.4	1.6	1.5	0.9	1.1
2.5	4.1	3.7	**81.1**	4.0	2.3	2.1	1.3	1.5
3	2.8	3.1	3.4	**81.7**	2.8	2.8	1.4	1.9
3.5	1.5	2.0	2.9	4.7	**79.9**	4.2	2.2	2.6
4	1.1	1.1	1.7	3.7	3.3	**82.2**	3.7	3.1
4.5	0.7	1.0	1.1	2.0	2.9	3.7	**83.6**	5.0
Stressed Migration Matrix								
Ratings	1.5	2	2.5	3	3.5	4	4.5	Default
1.5	**79.5**	5.1	3.8	3.4	2.1	2.7	1.5	1.9
2	2.5	**76.1**	5.2	5.7	2.9	2.8	2.0	2.8
2.5	1.6	1.8	**76.4**	6.3	3.8	3.8	2.5	3.7
3	1.0	1.4	1.8	**78.8**	4.6	5.0	2.8	4.5
3.5	0.5	0.8	1.4	2.5	**77.8**	7.0	4.1	5.9
4	0.4	0.5	0.8	1.8	1.8	**81.4**	6.5	6.9
4.5	0.2	0.4	0.4	0.9	1.5	2.0	**84.3**	10.3

Using the Duration migration matrices, the table reports the conditional transition migration matrices under the stress events described in section 4.5.2. The baseline migration matrix without stress scenario is also presented for comparison.

Table 4.17: Simulated Portfolio Stressed Economic Capital

Cohort		Duration	
Baseline scenario	Stress scenario	Baseline scenario	Stress scenario
15,521	28,046	16,072	26,557
23,147	35,574	23,784	34,174

Stressed credit risk loss for a one-year horizon is computed by a one-factor version of CreditMetrics approach described in section 4.5.1. The results are based on 2008Q3 migration matrix via criteria D1 for the Cohort and Duration estimates. The sample portfolio is described in Table 4.13. Monte Carlo simulations (100,000 trials) are used to obtain the portfolio value distributions one year hence. Economic Capital (EC) are calculated at 99% and 99.9 % percentile.

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